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NONLINEAR PERTURBATION OF m-ACCRETIVE OPERATORS

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ABSTRACT. Let X be a reflexive Banach space. Conditions sufficient to guarantee that the sum $A+B$, of two m-accretive operators A and B is m-accretive are provided. The basic requirements are that the operator B be bounded in some sense relative to A and that A and B be weakly closed.

The object of this paper is the investigation of the additive perturbation of a m-accretive, possibly nonlinear, operator A by a m-accretive, possibly nonlinear, operator B. Our basic requirements are that the Banach space be reflexive and that the operator B be bounded in some sense relative to A. Our results relate to the work of T. Kato **[8]** and J. Mermin **[12]** concerning perturbation in Banach spaces which have uniformly convex dual. For recent work concerning perturbation the reader is referred to G. F. Webb **[14]**, **[15]**, V. Barbu **[1]**, J. Goldstein **[6]** and Y. Konishi **[11]**.

In what follows X will be a reflexive Banach space with norm $\|\cdot\|$; X^* will denote the dual space of X, and the pairing between X and X^* will be denoted by $\langle\ ,\ \rangle$. If A and B are operators mapping subsets of X to X we define the sum $A+B$ by the equation $(A+B)x=Ax+Bx$ for $x\in D(A)\cap D(B)$.

DEFINITION 1.1. Let A be a nonlinear operator mapping a subset of a Banach space X to X. A is said to be *accretive* provided that $\|x+\lambda Ax-(y+\lambda Ay)\|\geqq\|x-y\|$ for all $\lambda\geqq 0$ and $x,y\in D(A)$. An accretive operator is said to be *m-accretive* provided that $R(I+\lambda A)=X$ for all $\lambda\geqq 0$.

T. Kato **[9]** has shown that the definition of accretiveness is equivalent to the statement that $\operatorname{Re}\langle Ax-Ay,f\rangle\geqq 0$ for $x,y\in D(A)$ and some $f\in F(x-y)$ where F is the duality map from X to X^*. If A is an m-accretive operator, A has no proper accretive extension. However not every maximal accretive operator is m-accretive. If A is accretive and $n\in Z^+$ we define

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$J_n x=(I+1/nA)^{-1}x$ for $x \in D_n=R(I+1/nA)$. We define the Yosida approximations $A_n x=n(I-J_n)x$. The following facts are well known:

$$
\begin{aligned}
\|J_n x - J_n y\| &\leqq \|x - y\| && \text{for } x, y \in D_n,\\
\|J_n x - x\| &\leqq n^{-1}\,\|Ax\| && \text{for } x \in D(A) \cap D_n,\\
A_n x &= AJ_n x && \text{for } x \in D_n,\\
\|A_n x\| &\leqq \|Ax\| && \text{for } x \in D(A) \cap D_n.
\end{aligned}
\tag{1.1}
$$

If A is an m-accretive operator then A_n is an everywhere defined Lipschitz continuous accretive operator with Lipschitz constant $2n$.

Henceforth we shall use the symbol "$\to$" to denote strong convergence in X and "$\rightharpoonup$" to denote weak convergence in X.

DEFINITION 1.2. Let A be a nonlinear operator mapping a subset of X to X. A is said to be *weakly closed* provided that $\{x_n\}\subseteq D(A)$, $x_n \rightharpoonup x$, and $Ax_n \rightharpoonup y$ imply that $x \in D(A)$ and $Ax=y$.

If X is a reflexive Banach space and A is a weakly closed operator then A has the following property: if $\{x_n\}\subseteq D(A)$, $x_n \rightharpoonup x$ and $\|Ax_n\|\leqq M$ for some $M>0$ then $Ax_n \rightharpoonup Ax$. Let us refer to this property of operators as *condition W*. We now specify the type of perturbation to be considered.

DEFINITION 1.3. Let A and B be nonlinear operators defined on subsets of a Banach space X. If $D(A)\subseteq D(B)$ then B is said to be a perturbation of A of *type K* provided that there exist constants $a<1$, $b\geqq 0$ and $c\geqq 0$ so that,

$$\|Bx\| \leqq a\,\|Ax\| + b\,\|x\| + c \quad \text{for } x \in D(A). \tag{1.2}$$

We now make precise our notion of strong solutions to the Cauchy initial value problem.

DEFINITION 1.4. Let A be an operator defined on a subset of a Banach space X. By a strong solution to the Cauchy initial value problem,

$$u'(t) + Au(t) = 0; \qquad u(0) = x,\ t \in [0, T), \tag{1.3}$$

we mean a function $u\colon [0, T)\to X$ such that u is Lipschitz continuous on compact subsets of $[0, T)$; $u(0)=x$; $u'(t)$ exists and satisfies (1.3) for a.e. $t \in [0, T)$.

We now provide conditions sufficient for the sum of two operators to be accretive.

LEMMA 1.1. *Let A and B be nonlinear accretive operators on a Banach space X. If B is m-accretive and satisfies condition W then $A+B$ is accretive.*

PROOF. Let $x \in D(B)$ then condition W and equations (1.1) imply that $B_n x=n(I-(I+1/nB)^{-1})x$ converges weakly to Bx. Since B is m-accretive, B_n is accretive and everywhere defined. Moreover we can make use of the equivalent formulation of accretiveness to observe that B_n

is strongly accretive, i.e. $\mathrm{Re}\langle B_n x - B_n y, f\rangle \geqq 0$ for all $f \in F(x-y)$. The accretiveness of A implies that for each $x, y \in D(A)$ there is an $f' \in F(x-y)$ so that $\langle Ax - Ay, f'\rangle \geqq 0$. If we assume that $x, y \in D(A) \cap D(B)$ we can conclude that

$$\mathrm{Re}\langle (A + Bx) - (A + B)y, f'\rangle = \lim\langle (A + B_n)x - (A + B_n)y, f'\rangle \geqq 0.$$

We now proceed to establish a global existence theorem for Cauchy problems involving the operator $A+B$.

THEOREM 1. *Let A and B be weakly closed m-accretive operators defined on a reflexive Banach space X. If B is a perturbation of type K of A then for each $x \in D(A)$ there exists a unique global solution to the Cauchy initial value problem*

$$(1.4) \quad u'(t) + (A + B)u(t) = 0; \qquad u(0) = x \quad \textit{for a.e. } t \in [0, \infty).$$

PROOF. We proceed by picking an arbitrary $T < \infty$ and demonstrating that (1.4) has a unique strong solution on $[0, T]$. Denoting $B_n x = n(I - (I + 1/nB)^{-1})x$, we consider the operators $A + B_n$. A recent result of V. Barbu [**1**] together with the m-accretiveness of A and the continuity of B_n insures that $A + B_n$ is m-accretive. We now consider the approximate Cauchy problems

$$(1.5) \qquad u_n'(t) + (A + B_n)u_n(t) = 0; \qquad u(0) = x \quad \text{for } t \in [0, T].$$

The m-accretiveness of $A + B_n$ together with recent results of Crandall and Liggett [**3**] and Brezis and Pazy [**2**] guarantee that for each $n \in Z^+$, (1.5) has a strong solution; moreover each $u_n(t)$ may be represented as the product integral $u_n(t) = \lim_{m\to\infty}(I + t(A + B_n)/m)_x^{-m}$ uniformly for $t \in [0, T]$. Repeated applications of the fourth assertion of (1.1) yield

$$\|(A + B_n)(I + t(A + B_n n)/m)_x^{-m}\| \leqq \|Ax\| + \|Bx\|.$$

We observe that condition W implies that $(A + B_n)(I + t(A + B_n)/m)_x^{-m} \rightharpoonup (A + B_n)u_n(t)$ and thereby insures the existence of a constant $M > 0$ so that

$$(1.6) \qquad \begin{aligned} \|u_n(t) - u_n(\tau)\| &\leqq |t - \tau|\, M \quad \text{for } t, \tau \in [0, T]; \\ \|u_n'(t)\| = \|(A + B_n)u_n(t)\| &\leqq M \quad \text{for a.e. } t \in [0, T]. \end{aligned}$$

We now claim that there is a subsequence $\{u_{n'}(t)\}$ of $\{u_n(t)\}$ which converges weakly to a function $u(t)$ which also satisfies the Lipschitz condition (1.6) with constant M. The argument of Lemma 2.1 [**5**] is directly applicable to establish this convergence. We relabel the weakly convergent subsequence as $\{u_n(t)\}$.

We now seek to ascertain that $(A + B_n)u_n(t) \rightharpoonup (A + B)u(t)$. Since $u_n(t) \in D(A)$, $u_n(t) \in D(B)$; using properties of perturbations of type K

and statements (1.1) and (1.6) we have

$$(1.7)\quad \begin{aligned}(1-a)\,\|Au_n(t)\| - b\,\|u_n(t)\| - c &\leqq \|Au_n(t)\| - \|Bu_n(t)\| \\ &\leqq \|Au_n(t)\| - \|B_n u_n(t)\| \leqq M.\end{aligned}$$

Because $a<1$ we can conclude that there is a constant M' so that

$$(1.8)\quad \begin{aligned}\|Au_n(t)\| &\leqq M' \quad \text{for } t \in [0, T], \\ \|B_n u_n(t)\| &\leqq M' \quad \text{for } t \in [0, T].\end{aligned}$$

That $Au_n(t) \rightharpoonup Au(t)$ follows from the fact that weakly closed operators satisfy condition W. To see that $B_n u_n(t) \rightharpoonup Bu(t)$ we recall that $B_n x = B(I+n^{-1}B)_x^{-1}$; use statement (1.1) to see that $(I+n^{-1}B)^{-1}u_n(t) \rightharpoonup u(t)$ and apply condition W.

Since $u_n(t)$ is a strong solution to (1.5) we have the following equation:

$$(1.9)\quad \langle u_n(t), f\rangle = \langle x, f\rangle - \int_0^t \langle (A + B_n)u_n(s), f\rangle\, ds \qquad \text{for } x \in D(A), f \in X^*, t \in [0, T].$$

Taking the limit as $n\to\infty$ of each side of (1.9) we obtain $\langle u(t), f\rangle = \langle x, f\rangle - \int_0^t \langle (A + B)u(s), f\rangle\, ds$, and hence deduce that

$$u(t) = x - \int_0^t (A + B)u(s)\, ds \quad \text{for } t \in [0, T].$$

The above integral can be differentiated and we obtain $u'(t) + (A+B)u(t)=0$ for a.e. $t \in [0, T]$ and $u(0)=x$. The uniqueness of solutions to (1.4) on $[0, T]$ follows from the accretiveness of $A+B$ and standard techniques, cf. [**9**].

The next lemma connects the m-accretiveness of an accretive operator with the existence of strong solutions to a Cauchy initial value problem, cf. Kato [**8**].

LEMMA 1.2. *Let X be a Banach space and A be a closed nonlinear accretive operator. Then A is m-accretive provided that there exists an $x \in D(A)$ such that for all $p \in X$ the Cauchy initial value problem*

$$(1.10)\quad u'(t) + (A + I)u(t) - p = 0; \qquad u(0) = x, \quad t \in [0, \infty),$$

has a strong solution.

PROOF. Let $x \in D(A)$ satisfy the hypothesis of the lemma; we shall use the existence of a solution to (1.10) to show that there is a $v \in D(A)$ so that $v=\lim_{t\to\infty} u(t)$ and $(A+I)v-p=0$. Thus $R(I+A)=X$ and we can refer to a result of Oharu [**13**] to guarantee that $R(I+\lambda A)=X$ for all $\lambda \geqq 0$.

Utilizing the accretiveness of A and standard techniques we see that for a.e. $t, h \in [0, \infty)$ and $f \in F(u(t+h)-u(t))$,

$$(d/dt)\,\|u(t+h)-u(t)\|^2 = -2\,\mathrm{Re}\langle (A+I)u(t+h)-(A+I)u(t), f\rangle$$
$$\leqq -2\,\|u(t+h)-u(t)\|^2,$$

and hence that

$$(d/dt)\{e^{2t}\,\|u(t+h)-u(t)\|^2\} \leqq 0. \tag{2.11}$$

We integrate (2.11) on $(0, t)$ to obtain the inequality

$$\|u(t+h)-u(t)\| \leqq e^{-t}\,\|u(h)-u(0)\|.$$

Since u is a strong solution to the Cauchy problem we can conclude that there is a $M>0$ so that

$$\|u'(t)\| \leqq e^{-t}M \quad \text{for a.e. } t \in [0, \infty). \tag{2.12}$$

Since $\|u(t+h)-u(t)\| \leqq \int_t^{t+h} \|u'(s)\|\,ds$ we use (2.12) to conclude that there is a $v = \lim_{t\to\infty} u(t)$. Let $\{t_i\}_{i=1}^{\infty}$ be an increasing sequence of numbers at which equation (1.10) is satisfied such that $\lim t_i = \infty$. By virtue of (2.12) we have $\lim \|(A+I)u(t_i)-p\| = 0$ and thus we can invoke the closedness of A to establish $(A+I)v = p$.

We are now in a position to state and prove our principal result.

THEOREM 2. *Let X be a reflexive Banach space and let A and B be nonlinear, weakly closed m-accretive operators such that $D(A) \subseteq D(B)$. If B is a perturbation of A of type K then $A+B$ is m-accretive.*

PROOF. Theorem 2 is obtained by defining $B_p x = (B+I)x - p$. Clearly if B satisfies the hypotheses of Theorems 1 and 2 then so does B_p. Thus Theorem 2 follows from Theorem 1 by immediate application of Lemma 1.2.

If A is linear we have the following corollary:

COROLLARY. *Let A be a closed linear m-accretive operator defined on a reflexive Banach space X. If B is a nonlinear, weakly closed, m-accretive perturbation of A of type K, then $A+B$ is m-accretive.*

PROOF. We need only observe that a strongly closed linear operator is weakly closed.

If we further require that the operator B be everywhere defined and weakly continuous, i.e., that B map weakly convergent sequences to weakly convergent sequences, we can eliminate the requirement that B be bounded relative to A.

THEOREM 3. *Let A be a closed, linear m-accretive operator on a reflexive Banach space. If B is a nonlinear, weakly continuous accretive operator then $A+B$ is m-accretive.*

PROOF. In [4] it has been shown that a weakly continuous accretive operator on a reflexive Banach space is m-accretive. Following the proof of Theorem 1 we guarantee the existence of unique solutions to the approximate equations $u_n'(t)+(A+B_n)u_n(t)=0$; $u_n(0)=x$, $t \in [0, T]$.

As before we obtain the existence of $M>0$ so that $\|u_n(t)-u_n(\tau)\| \leqq |t-\tau|M$ and $\|(A+B_n)u_n(t)\| \leqq M$. We now observe that weakly continuous operators in reflexive spaces map bounded subsets to bounded subsets. Since $\|(I+n^{-1}B)^{-1}u_n(t)-u_n(t)\| \leqq n^{-1}\|Bu_n(t)\|$ and $\|B_nu_n(t)\| \leqq \|Bu_n(t)\|$ we see that $\|B_nu_n(t)\|$ and $\|Au_n(t)\|$ are bounded independently of n. We are now able to apply the remainder of the previous argument.

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