

NONIMMERSIONS OF LOW DIMENSIONAL FLAG MANIFOLDS

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ABSTRACT. Certain useful quadratic identities in the cohomology of classifying spaces induce quadratic equations in the cohomology of a manifold M under the classifying map for the normal bundle of M . In low dimensional flag manifolds, one can show that the quadratic equation has no root, thus establishing a nonimmersion.

1. Introduction. Let $BSO(r)$ be the classifying space for oriented real r -plane bundles. In integral cohomology, one has the quadratic identity

$$(1.1) \quad p_r = \chi^2 \in H^{4r}(BSO(2r); Z),$$

where p_r is the r th Pontryagin class and χ is the Euler class. By a well-known result of Hirsch [6], an oriented manifold M immerses in codimension k iff the stable normal bundle ν has dimension k ; i.e., ν is classified by a map $f_\nu: M \rightarrow BSO(k)$. Thus, if k is even, identity (1.1) induces a quadratic equation in $H^*(M; Z)$. One obtains nonimmersion by showing that this equation has no root.

We shall restrict our attention to complex flag manifolds,

$$F(n_1, n_2, \dots, n_k) = U(n)/U(n_1) \times U(n_2) \times \dots \times U(n_k),$$

where $n = n_1 + n_2 + \dots + n_k$. We shall show that

- (a) If $M = F(2, 2)$, then $\dim M = 8$, $M \subseteq \mathbf{R}^{14}$, $\not\subseteq \mathbf{R}^{12}$.
- (b) If $M = F(2, 3)$, then $\dim M = 12$, $M \subseteq \mathbf{R}^{23}$, $\not\subseteq \mathbf{R}^{19}$.

The methods of this paper also show that

- (c) If $M = F(1, 2, 2)$, then $\dim M = 16$, $M \subseteq \mathbf{R}^{22}$, $\not\subseteq \mathbf{R}^{20}$.
- (d) If $M = F(2, 2, 2)$, then $\dim M = 24$, $M \subseteq \mathbf{R}^{33}$, $\not\subseteq \mathbf{R}^{30}$.

The methods of this paper were first used by Feder [5] to obtain non-immersions of projective spaces CP^n , n odd. Mayer [9] obtains very general

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nonimmersion results for manifolds, but our methods give stronger results for the flag manifolds considered here.

2. The adjoint representation. The normal bundle of any homogeneous space, and in particular, of flag manifolds, is determined by the adjoint representation. Let G be a Lie group and H a closed subgroup. Let π denote the H -bundle $H \rightarrow G \xrightarrow{\pi} G/H$ where $G \rightarrow G/H$ is the canonical projection onto the space of right cosets of H in G . The space G/H is called a homogeneous space. The mixing construction $\alpha_\pi: RO(H) \rightarrow KO(G/H)$ from real representation theory to K -theory is a ring morphism defined by letting $\alpha_\pi[\rho] \in KO(G/H)$ be the bundle

$$(2.1) \quad G/H \xrightarrow{f_\pi} BH \xrightarrow{B\rho} BO(n),$$

where f_π classifies the bundle π over G/H , and $\rho: H \rightarrow O(n)$ is a representation.

Borel and Hirzebruch [4, (1.1)] have determined the tangent bundle $\tau_{G/H}$ of any homogeneous space G/H to be

$$\tau_{G/H} = \alpha_\pi(\text{Ad}_G|_H - \text{Ad}_H) \in KO(G/H).$$

Here Ad_H is the adjoint representation of H and $\text{Ad}_G|_H$ is the restriction of Ad_G to H . Note that by (2.1), $\alpha_\pi(\text{Ad}_G|_H)$ is the bundle

$$G/H \xrightarrow{f_\pi} BH \xrightarrow{Bi} BG \xrightarrow{BAa} BO(g),$$

where $i: H \rightarrow G$ is the inclusion and $g = \dim G$. Since $Bi \circ f_\pi$ is homotopically trivial, $\alpha_\pi(\text{Ad}_G|_H) = g$. Hence, we have

$$\tau_{G/H} = g - \alpha_\pi(\text{Ad}_H).$$

Thus, the normal bundle $\nu_{G/H} = \dim G/H - \tau_{G/H}$ is given by

$$(2.2) \quad \nu_{G/H} = \alpha_\pi(\text{Ad}_H) - h,$$

where $h = \dim H$.

3. Cohomology of the flag manifolds. In this section, we compute both the integral cohomology and complex K -theory cohomology of the flag manifolds. We adopt the following notation. Let $G = U(n)$, $H = U(n_1) \times \cdots \times U(n_k)$, thus the flag manifold $F = F(n_1, \dots, n_k)$ is given by $F = G/H$. Denote by $w: H \rightarrow G$ the usual inclusion map given by direct sum of matrices, and let $f_\pi: F = G/H \rightarrow BH$ be the classifying map for the principal H -bundle $H \rightarrow G \xrightarrow{\pi} F = G/H$.

Let $i: p \rightarrow X$ be the inclusion of a point, and define $\bar{H}(X) = \text{Ker } i^*: H^*(X) \rightarrow H^*(p)$. Then we have the following result of Borel [3].

PROPOSITION 3.1. *The integral cohomology of $F=F(n_1, \dots, n_k)$ is torsion-free and is determined by the following short exact sequence of rings:*

$$0 \longrightarrow \bar{H}(BG; Z) \cdot H^*(BH; Z) \longrightarrow H^*(BH; Z) \xrightarrow{f_\pi^*} H^*(F; Z) \longrightarrow 0.$$

Here, the term $\bar{H}(BG) \cdot H^*(BH)$ is the ideal of $H^*(BH)$ generated by $\bar{H}(BG)$.

We explicitly compute generators and relations for the cohomology of $F(2, 2)=U(4)/U(2) \times U(2)$. If $p(X, t)$ denotes the Poincaré power series $p(x, t)=\sum_{i \geq 0} \text{rank } H^i(X; Z)t^i$ for any space X , then by [8], we have

$$(3.2) \quad \begin{aligned} p(F(2, 2), t) &= p(BU(2) \times BU(2), t)/p(BU(4), t) \\ &= 1 + t^2 + 2t^4 + t^6 + t^8. \end{aligned}$$

If we now define $x_i \in H^{2i}(F(2, 2); Z)$, $i=1, 2$, by $x_i=f_\pi^*(a_i)$, where $a_i=c_i \otimes 1 \in H^*(BU(2); Z) \otimes H^*(BU(2); Z)$ is the Chern class of the first factor in $BU(2) \times BU(2)$ and $f_\pi: F(2, 2) \rightarrow BU(2) \times BU(2)$ the usual map, we have

PROPOSITION 3.3. *The cohomology ring $H^*(F(2, 2); Z)$ is torsion-free of rank 6 with additive generators $1 \in H^0$, $x_1 \in H^2$, x_1^2 and $x_2 \in H^4$, $x_1x_2 \in H^6$, and $x_1^2x_2 \in H^8$.*

REMARK. We have, for example, $x_1^3=2x_1x_2$.

PROOF. From (3.1) and (3.2), we know that $H^*(F(2, 2), Z)$ is torsion-free of rank 6. The six elements listed additively generate the image of $f_\pi^*: H^*(BU(2) \times BU(2)) \rightarrow H^*(F(2, 2))$. Since f_π^* is onto by (3.1), we are done.

Letting $\alpha_i=\gamma_i \otimes 1 \in RU(2) \otimes RU(2)$, $i=1, 2$ where γ_i is the K -theory Chern class (see [1, §2]), we define $\xi_i=\alpha_\pi(\alpha_i) \in KF(2, 2)$, $i=1, 2$, where $\alpha_\pi: RU(2) \times RU(2) \rightarrow KF(2, 2)$ is the mixing construction.

PROPOSITION 3.4. *The K -theory ring $KF(2, 2)$ is torsion-free of rank 6 with additive generators $1, \xi_1, \xi_1^2, \xi_2, \xi_1\xi_2$, and $\xi_1^2\xi_2$. Furthermore, the ring structure of $KF(2, 2)$ is formally the same as that of $H^*(F(2, 2); Z)$.*

PROOF. The Chern character $\text{ch}: KF \rightarrow H^{**}(F; C)$ is calculated to be

$$\begin{aligned} \text{ch}(\xi_1) &= x_1 + \frac{1}{2}(x_1^2 - 2x_2) - \frac{1}{6}x_1x_2, \\ \text{ch}(\xi_2) &= x_2 + \frac{1}{2}x_1x_2 + \frac{1}{12}x_1^2x_2, \end{aligned}$$

where the x_i are the generators of (3.3). The proof now follows immediately from [2, §2.5].

4. The quadratic equation. We now apply identity (1.1) to obtain non-immersion of $F(2, 2)=U(4)/U(2) \times U(2)$ in codimension 4. If $F(2, 2)$ immerses in codimension 4, its stable normal bundle ν is an $SO(4)$ bundle.

Letting $r=2$, identity (1.1) induces a quadratic equation $p_2(v)=\chi^2 \in H^8(F(2, 2); Z)$ for some element χ . Using (2.2), one calculates $p_2(v)=-6x_1^2x_2$, where $x_1^2x_2$ is the generator of H^8 given in (3.3). Solving for χ , we write $\chi=ax_2+bx_1^2 \in H^4$ and obtain $-6=(a+b)^2+b^2$. There are no integer solutions, and thus nonimmersion in codimension 4 is established. Tornehave [10] has established positive immersion results for flag manifolds. Thus, we have

PROPOSITION 4.1. *The eight dimensional manifold $F(2, 2)$ does not immerse in codimension 4 but does immerse in codimension 6.*

The K -theoretic methods of [1] and [5] give no new results for $F(2, 2)$.

5. Another quadratic equation. Turning attention to the manifold $F(2, 3)$, we recall that $\text{Spin}(k)$ is the double cover of $SO(k)$, and that $\text{Spin}^c(k)$ is defined by the following pullback diagram:

$$\begin{array}{ccc} \text{Spin}^c(k) & \longrightarrow & \text{Spin}(2+k) \\ \downarrow & & \downarrow \\ SO(2) \times SO(k) & \longrightarrow & SO(2+k) \end{array}$$

Here, $SO(2) \times SO(k) \rightarrow SO(2+k)$ is the standard inclusion.

For any group G , let RG denote the ring of unitary representations. Following [7], we define an element $\tilde{\Delta} \in R \text{Spin}^c(2k+\epsilon)$, $\epsilon=0$ or 1 , by $\tilde{\Delta}=z_0^{-1/2} \prod_{i=1}^k (z_i^{1/2} + z_i^{-1/2})$, where z_0 is the basic toral representation of $SO(2)$ and $z_i, i=1, \dots, k$, are the toral representations of $SO(2k+\epsilon)$. One calculates that $\tilde{\Delta}$ satisfies

$$(5.1) \quad a_\epsilon \tilde{\Delta}^2 = z_0 \cdot \lambda_1 \in R \text{Spin}^c(2k+\epsilon), \quad \epsilon = 0 \text{ or } 1,$$

where $a_0=1, a_1=2$ and λ_1 is the sum of the exterior power elements $\lambda^i = \sigma_i(z_1, z_1^{-1}, \dots, z_k, z_k^{-1}, (1))$, $i=0, 1, \dots, 2k+\epsilon$, $\sigma_i=i$ th symmetric function.

Since the third integral Stiefel-Whitney class W_3 of $F(2, 3)$ is zero, we know [7, (5.5)] that $F(2, 3)$ is a Spin^c manifold. In fact, by an appropriate choice of an $SO(2)$ bundle δ over $F(2, 3)$, one can factor the normal bundle $F(2, 3) \xrightarrow{\nu} BSO(n)$ as follows:

$$\begin{array}{ccc} & & B \text{Spin}^c(n) \\ & \nearrow^{\delta+\nu} & \downarrow B_\rho \\ & & BSO(2) \times BSO(n) \\ & \nearrow^{\delta+\nu} & \downarrow P_2 \\ F(2, 3) & \xrightarrow{\nu} & BSO(n) \end{array}$$

where ρ is the double cover and p_2 is projection on the second factor.

If now, $F(2, 3)$ immerses, in codimension 7, the above diagram gives a mapping $F(2, 3) \xrightarrow{\theta+\nu} B \text{Spin}^c(7)$ factoring the normal bundle $F(2, 3) \xrightarrow{\nu} BSO(7)$. Thus identity (5.1) induces a quadratic equation in $KF(2, 3)$. One calculates $KF(2, 3)$ as in (3.4), and shows that the coefficient of the $\xi_1^2 \xi_2$ term in $\tilde{\Delta}$ is $7/2$. Thus, there is no integral solution for $\tilde{\Delta}$, and non-immersion in codimension 7 is established. A positive immersion result for $F(2, 3)$ follows from [10]. Thus, we have

PROPOSITION 5.2. *The 12-dimensional manifold $F(2, 3)$ does not immerse in codimension 7, but does immerse in codimension 11.*

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