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NONIMMERSIONS OF LOW DIMENSIONAL FLAG MANIFOLDS

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ABSTRACT. Certain useful quadratic identities in the cohomology of classifying spaces induce quadratic equations in the cohomology of a manifold M under the classifying map for the normal bundle of M. In low dimensional flag manifolds, one can show that the quadratic equation has no root, thus establishing a nonimmersion.

1. **Introduction.** Let $BSO(r)$ be the classifying space for oriented real r-plane bundles. In integral cohomology, one has the quadratic identity

$$p_r = \chi^2 \in H^{4r}(BSO(2r); Z), \tag{1.1}$$

where p_r is the rth Pontryagin class and χ is the Euler class. By a well-known result of Hirsch [**6**], an oriented manifold M immerses in codimension k iff the stable normal bundle ν has dimension k; i.e., ν is classified by a map $f_\nu : M \to BSO(k)$. Thus, if k is even, identity (1.1) induces a quadratic equation in $H^*(M; Z)$. One obtains nonimmersion by showing that this equation has no root.

We shall restrict our attention to complex flag manifolds,

$$F(n_1, n_2, \cdots, n_k) = U(n)/U(n_1) \times U(n_2) \times \cdots \times U(n_k),$$

where $n = n_1 + n_2 + \cdots + n_k$. We shall show that

(a) If $M = F(2, 2)$, then dim $M = 8$, $M \subseteq \boldsymbol{R}^{14}$, $\nsubseteq \boldsymbol{R}^{12}$.

(b) If $M = F(2, 3)$, then dim $M = 12$, $M \subseteq \boldsymbol{R}^{23}$, $\nsubseteq \boldsymbol{R}^{19}$.

The methods of this paper also show that

(c) If $M = F(1, 2, 2)$, then dim $M = 16$, $M \subseteq \boldsymbol{R}^{22}$, $\nsubseteq \boldsymbol{R}^{20}$.

(d) If $M = F(2, 2, 2)$, then dim $M = 24$, $M \subseteq \boldsymbol{R}^{33}$, $\nsubseteq \boldsymbol{R}^{30}$.

The methods of this paper were first used by Feder [**5**] to obtain nonimmersions of projective spaces CP^n, n odd. Mayer [**9**] obtains very general

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nonimmersion results for manifolds, but our methods give stronger results for the flag manifolds considered here.

2. **The adjoint representation.** The normal bundle of any homogeneous space, and in particular, of flag manifolds, is determined by the adjoint representation. Let G be a Lie group and H a closed subgroup. Let π denote the H-bundle $H \to G \xrightarrow{\pi} G/H$ where $G \to G/H$ is the canonical projection onto the space of right cosets of H in G. The space G/H is called a homogeneous space. The mixing construction $\alpha_\pi : RO(H) \to KO(G/H)$ from real representation theory to K-theory is a ring morphism defined by letting $\alpha_\pi[\rho] \in KO(G/H)$ be the bundle

$$G/H \xrightarrow{f_\pi} BH \xrightarrow{B\rho} BO(n), \tag{2.1}$$

where f_π classifies the bundle π over G/H, and $\rho : H \to O(n)$ is a representation.

Borel and Hirzebruch [**4**, (1.1)] have determined the tangent bundle $\tau_{G/H}$ of any homogeneous space G/H to be

$$\tau_{G/H} = \alpha_\pi(\mathrm{Ad}_G|H - \mathrm{Ad}_H) \in KO(G/H).$$

Here Ad_H is the adjoint representation of H and $\mathrm{Ad}_G|H$ is the restriction of Ad_G to H. Note that by (2.1), $\alpha_\pi(\mathrm{Ad}_G|H)$ is the bundle

$$G/H \xrightarrow{f_\pi} BH \xrightarrow{Bi} BG \xrightarrow{BAd} BO(g),$$

where $i : H \to G$ is the inclusion and $g = \dim G$. Since $Bi \circ f_\pi$ is homotopically trivial, $\alpha_\pi(\mathrm{Ad}_G|H) = g$. Hence, we have

$$\tau_{G/H} = g - \alpha_\pi(\mathrm{Ad}_H).$$

Thus, the normal bundle $\nu_{G/H} = \dim G/H - \tau_{G/H}$ is given by

$$\nu_{G/H} = \alpha_\pi(\mathrm{Ad}_H) - h, \tag{2.2}$$

where $h = \dim H$.

3. **Cohomology of the flag manifolds.** In this section, we compute both the integral cohomology and complex K-theory cohomology of the flag manifolds. We adopt the following notation. Let $G = U(n)$, $H = U(n_1) \times \cdots \times U(n_k)$, thus the flag manifold $F = F(n_1, \cdots, n_k)$ is given by $F = G/H$. Denote by $w : H \to G$ the usual inclusion map given by direct sum of matrices, and let $f_\pi : F = G/H \to BH$ be the classifying map for the principal H-bundle $H \to G \xrightarrow{\pi} F = G/H$.

Let $i : p \to X$ be the inclusion of a point, and define $\bar{H}(X) = \mathrm{Ker}\, i^* : H^*(X) \to H^*(p)$. Then we have the following result of Borel [**3**].

PROPOSITION 3.1. *The integral cohomology of* $F=F(n_1, \cdots, n_k)$ *is torsion-free and is determined by the following short exact sequence of rings*:

$$O \longrightarrow \bar{H}(BG;Z)\cdot H^*(BH;Z) \longrightarrow H^*(BH;Z) \xrightarrow{f_\pi^*} H^*(F;Z) \longrightarrow 0.$$

Here, the term $\bar{H}(BG)\cdot H^*(BH)$ *is the ideal of* $H^*(BH)$ *generated by* $\bar{H}(BG)$.

We explicitly compute generators and relations for the cohomology of $F(2,2)=U(4)/U(2)\times U(2)$. If $p(X,t)$ denotes the Poincaré power series $p(x,t)=\sum_{i\geqq 0}\operatorname{rank} H^i(X;Z)t^i$ for any space X, then by [**8**], we have

$$\begin{aligned} p(F(2,2),t) &= p(BU(2)\times BU(2),t)/p(BU(4),t) \\ &= 1+t^2+2t^4+t^6+t^8. \end{aligned} \tag{3.2}$$

If we now define $x_i\in H^{2i}(F(2,2);Z)$, $i=1,2$, by $x_i=f_\pi^*(a_i)$, where $a_i=c_i\otimes 1\in H^*(BU(2);Z)\otimes H^*(BU(2);Z)$ is the Chern class of the first factor in $BU(2)\times BU(2)$ and $f_\pi: F(2,2)\to BU(2)\times BU(2)$ the usual map, we have

PROPOSITION 3.3. *The cohomology ring* $H^*(F(2,2);Z)$ *is torsion-free of rank* 6 *with additive generators* $1\in H^0$, $x_1\in H^2$, x_1^2 *and* $x_2\in H^4$, $x_1x_2\in H^6$, *and* $x_1^2x_2\in H^8$.

REMARK. We have, for example, $x_1^3=2x_1x_2$.

PROOF. From (3.1) and (3.2), we know that $H^*(F(2,2),Z)$ is torsion-free of rank 6. The six elements listed additively generate the image of $f_\pi^*: H^*(BU(2)\times BU(2))\Rightarrow H^*(F(2,2))$. Since f_π^* is onto by (3.1), we are done.

Letting $\alpha_i=\gamma_i\otimes 1\in RU(2)\otimes RU(2)$, $i=1, 2$ where γ_i is the K-theory Chern class (see [**1**, §2]), we define $\xi_i=\alpha_\pi(\alpha_i)\in KF(2,2)$, $i=1,2$, where $\alpha_\pi: RU(2)\times RU(2)\to KF(2,2)$ is the mixing construction.

PROPOSITION 3.4. *The K-theory ring* $KF(2,2)$ *is torsion-free of rank* 6 *with additive generators* 1, ξ_1, ξ_1^2, ξ_2, $\xi_1\xi_2$, *and* $\xi_1^2\xi_2$. *Furthermore, the ring structure of* $KF(2,2)$ *is formally the same as that of* $H^*(F(2,2);Z)$.

PROOF. The Chern character $\operatorname{ch}: KF\to H^{**}(F;C)$ is calculated to be

$$\begin{aligned} \operatorname{ch}(\xi_1) &= x_1+\tfrac{1}{2}(x_1^2-2x_2)-\tfrac{1}{6}x_1x_2, \\ \operatorname{ch}(\xi_2) &= x_2+\tfrac{1}{2}x_1x_2+\tfrac{1}{12}x_1^2x_2, \end{aligned}$$

where the x_i are the generators of (3.3). The proof now follows immediately from [**2**, §2.5].

4. **The quadratic equation.** We now apply identity (1.1) to obtain non-immersion of $F(2,2)=U(4)/U(2)\times U(2)$ in codimension 4. If $F(2,2)$ immerses in codimension 4, its stable normal bundle ν is an $SO(4)$ bundle.

Letting $r=2$, identity (1.1) induces a quadratic equation $p_2(\nu)=\chi^2 \in H^8(F(2,2);Z)$ for some element χ. Using (2.2), one calculates $p_2(\nu)=-6x_1^2x_2$, where $x_1^2x_2$ is the generator of H^8 given in (3.3). Solving for χ, we write $\chi=ax_2+bx_1^2 \in H^4$ and obtain $-6=(a+b)^2+b^2$. There are no integer solutions, and thus nonimmersion in codimension 4 is established. Tornehave **[10]** has established positive immersion results for flag manifolds. Thus, we have

PROPOSITION 4.1. *The eight dimensional manifold* $F(2,2)$ *does not immerse in codimension* 4 *but does immerse in codimension* 6.

The K-theoretic methods of **[1]** and **[5]** give no new results for $F(2,2)$.

5. **Another quadratic equation.** Turning attention to the manifold $F(2,3)$, we recall that $\mathrm{Spin}(k)$ is the double cover of $SO(k)$, and that $\mathrm{Spin}^c(k)$ is defined by the following pullback diagram:

$$\begin{array}{ccc} \mathrm{Spin}^c(k) & \longrightarrow & \mathrm{Spin}(2+k) \\ \downarrow & & \downarrow \\ SO(2)\times SO(k) & \longrightarrow & SO(2+k) \end{array}$$

Here, $SO(2)\times SO(k)\to SO(2+k)$ is the standard inclusion.

For any group G, let RG denote the ring of unitary representations. Following [7], we define an element $\tilde{\Delta} \in R\,\mathrm{Spin}^c(2k+\varepsilon)$, $\varepsilon=0$ or 1, by $\tilde{\Delta}=z_0^{1/2}\prod_{i=1}^{k}(z_i^{1/2}+z_i^{-1/2})$, where z_0 is the basic toral representation of $SO(2)$ and z_i, $i=1,\cdots,k$, are the toral representations of $SO(2k+\varepsilon)$. One calculates that $\tilde{\Delta}$ satisfies

(5.1) $$a_\varepsilon\tilde{\Delta}^2 = z_0\cdot\lambda_1 \in R\,\mathrm{Spin}^c(2k+\varepsilon), \qquad \varepsilon = 0 \text{ or } 1,$$

where $a_0=1$, $a_1=2$ and λ_1 is the sum of the exterior power elements $\lambda^i=\sigma_i(z_1, z_1^{-1},\cdots,z_k,z_k^{-1},(1))$, $i=0,1,\cdots,2k+\varepsilon$, $\sigma_i=i$th symmetric function.

Since the third integral Stiefel-Whitney class W_3 of $F(2,3)$ is zero, we know [7, (5.5)] that $F(2,3)$ is a Spin^c manifold. In fact, by an appropriate choice of an $SO(2)$ bundle δ over $F(2,3)$, one can factor the normal bundle $F(2,3)\xrightarrow{\nu} BSO(n)$ as follows:

$$\begin{array}{ccc} & & B\,\mathrm{Spin}^c(n) \\ & \overset{\delta+\nu}{\nearrow} & \downarrow{\scriptstyle B_\rho} \\ & & BSO(2)\times BSO(n) \\ & \overset{\delta+\nu}{\nearrow} & \downarrow{\scriptstyle P_2} \\ F(2,3) & \xrightarrow{\ \nu\ } & BSO(n) \end{array}$$

where ρ is the double cover and p_2 is projection on the second factor.

If now, $F(2, 3)$ immerses, in codimension 7, the above diagram gives a mapping $F(2, 3) \xrightarrow{\delta+\nu} B\ \mathrm{Spin}^c(7)$ factoring the normal bundle $F(2, 3) \xrightarrow{\nu} BSO(7)$. Thus identity (5.1) induces a quadratic equation in $KF(2, 3)$. One calculates $KF(2, 3)$ as in (3.4), and shows that the coefficient of the $\xi_1^2\xi_2$ term in $\tilde{\Lambda}$ is 7/2. Thus, there is no integral solution for $\tilde{\Delta}$, and non-immersion in codimension 7 is established. A positive immersion result for $F(2, 3)$ follows from [**10**]. Thus, we have

PROPOSITION 5.2. *The* 12-*dimensional manifold* $F(2, 3)$ *does not immerse in codimension* 7, *but does immerse in codimension* 11.

BIBLIOGRAPHY

1. M. F. Atiyah, *Immersions and embeddings of manifolds*, Topology **1** (1962), 125–132. MR **26** #3080.

2. M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R.I., 1961, pp. 7–38. MR **25** #2617.

3. A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115–207. MR **14**, 490.

4. A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces*. I, Amer. J. Math. **80** (1958), 458–538. MR **21** #1586.

5. S. Feder, *Non-immersion theorems for complex and quaternionic projective spaces*, Bl. Soc. Mat. Mexicana (2) **11** (1966), 62–67. MR **38** #721.

6. M. W. Hirsch, *Immersions of manifolds*, Trans. Amer. Math. Soc. **93** (1959), 242–276. MR **22** #9980.

7. F. Hirzebruch, *A Riemann-Roch theorem for differentiable manifolds*, Paris Univ., Séminaire Bourbaki, 11ième année; 1958/59, Exposés 177, 1959. MR **28** #1091.

8. A. Liulevicius, *Characteristic classes and cobordism*. Part I, Aarhus Univ., Lecture Notes, Fall 1967.

9. K. H. Mayer, *Elliptische differentialoperatoren und ganzzahligkeitsätze für charakteristische Zahlen*, Topology **4** (1965), 295–313. MR **33** #6650.

10. J. Tornehave, *Immersions of complex flagmanifolds*, Math. Scand. **23** (1968), 22–26. MR **40** #4970.

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