PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

EQUIVARIANT IMMERSION AND IMBEDDING UP TO COBORDISM

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ABSTRACT. This paper determines the smallest possible integer r so that a given manifold with involution is equivariantly bordant to an immersed or imbedded submanifold of some $R^s \times R^r$ with involution $1 \times (-1)$.

1. **Introduction.** Being given a class $\alpha \in \mathfrak{N}_n^{Z_2}$ in the cobordism group of manifolds with unrestricted involution, Bix [1] has defined integers $\phi(\alpha)$ and $\psi(\alpha)$ by:

(a) $\phi(\alpha)$ is the smallest integer r for which there is a representative (M^n, T) of α which *immerses* equivariantly in $R^{s,r}$ for some s, where $R^{s,r} = R^s \times R^r$ with involution $1 \times (-1)$, and

(b) $\psi(\alpha)$ is the smallest integer r for which there is a representative (M^n, T) of α which *imbeds* equivariantly in $R^{s,r}$ for some s.

Being given α, the fixed data of α consists of classes $\alpha_j \in \mathfrak{N}_{n-j}(BO_j)$. Let $\eta(\alpha)$ be the smallest integer k such that each α_j lies in the image of $\mathfrak{N}_{n-j}(G_{j,k})$, where $G_{j,k}$ is the Grassmanian of j planes in R^k.

Note. If $\alpha_j = [F^{n-j}, \nu^j] \in \mathfrak{N}_{n-j}(BO_j)$, the condition that α_j lie in the image of $\mathfrak{N}_{n-j}(G_{j,k})$ is that all Stiefel-Whitney numbers involving $\bar{w}_i(\nu)$ vanish if $i > k-j$.

THEOREM. $\phi(\alpha) = \psi(\alpha) = \eta(\alpha)$.

2. **Proof of the Theorem.** Clearly $\phi(\alpha) \leqq \psi(\alpha)$ since an imbedding is an immersion, so it suffices to show that $\eta(\alpha) \leqq \phi(\alpha)$ and $\psi(\alpha) \leqq \eta(\alpha)$.

To see that $\eta(\alpha) \leqq \phi(\alpha)$, suppose (M^n, T) represents α and $f: (M^n, T) \to R^{s,r}$ is an immersion. Then on the $(n-j)$-dimensional component of the fixed set F^{n-j}, the normal bundle in M, ν^j, has a complement ρ of dimension $r-j$, for F^{n-j} immerses in $R^s \times O$ and ν^j is a subbundle of the pullback of the normal bundle of $R^s \times O$ in $R^s \times R^r$, which is a trivial r-plane bundle. Thus ν^j is classified by a map into $G_{j,r}$ and $\eta(\alpha) \leqq r$.

Received by the editors August 8, 1973 and, in revised form, September 11, 1973.
AMS (MOS) subject classifications (1970). Primary 57D85, 57D40.
[1] The author is indebted to the National Science Foundation for financial support during this work.

To see that $\psi(\alpha)\leqq\eta(\alpha)$, let α be given and represent α_j by maps $F^{n-j}\to G_{j,\eta(\alpha)}$. Then the F^{n-j} may be imbedded disjointly in R^s for some large s ($s>2n+1$ will suffice) and the normal bundle ν^j has a complement ρ of dimension $\eta(\alpha)-j$ so that $D(\nu^j)\subset D(\nu^j\oplus\rho)$ imbeds (fiberwise) in the trivial bundle $R^s\times R^{\eta(\alpha)}$ over R^s (in fact in the space $R^s\times D^{\eta(\alpha)}$). Letting N be the tubular neighborhood of the fixed data given by the union of the $D(\nu^j)$, ∂N imbeds equivariantly in $R^s\times S^{\eta(\alpha)-1}$ or $\partial N/Z_2$ imbeds in $R^s\times RP(\eta(\alpha)-1)$. The map $g:\partial N/Z_2\to R^s\times RP(\eta(\alpha)-1)$ bounds in $R^s\times RP(\infty)$, that being the condition that the collection of α_j come from some α, and $\mathfrak{N}_*(RP(i))\to\mathfrak{N}_*(RP(\infty))$ is monic, so g bounds. For s sufficiently large ($s>2n+1$ being sufficient) g bounds an imbedded manifold with boundary $h: W\to R^s\times RP(\eta(\alpha)-1)$, and taking double covers $\tilde{h}:\tilde{W}\to R^s\times S^{\eta(\alpha)-1}$ is an imbedding. Joining N and $\tilde{W}$ along their common boundary gives a closed manifold M^n, with the involution T induced by -1 in the fibers of $D(\nu^j)$ and the involution of the double cover on $\tilde{W}$, and with an imbedding $f:(M, T)\to R^{s,\eta(\alpha)}$ induced by $\tilde{h}$ and the imbedding of $D(\nu^j)$ in $R^s\times D^{\eta(\alpha)}$. Since the fixed data of (M^n, T) is given by the F^{n-j} and ν^j, (M^n, T) represents α, and so $\psi(\alpha)\leqq\eta(\alpha)$.

Note. Using a tubular neighborhood of $\partial N/Z_2$ in W, one may make f smooth.

3. **Remarks.** One should compare this with Bix's result which computes $\eta(\alpha)$ for α in a certain base for $\mathfrak{N}_*^{Z_2}$ over Z_2.

To describe the base, one lets $RP(n)$ have the involution

$$T([x_0, \cdots, x_n]) = [-x_0, x_1, \cdots, x_n],$$

and being given M with involution T, $\Gamma(M)$ is $S^1\times M$ with (z, m) identified with $(-z, Tm)$ and with involution induced by $(z, m)\to(\bar{z}, m)$, where $\bar{z}$ is the conjugate of the complex number z. The base then consists of all classes $\alpha=\Gamma^i RP(n_1)\times RP(n_2)\times\cdots\times RP(n_k)\times M^m$, with the diagonal action, M having trivial action and $[M]$ forming a base for $\mathfrak{N}_*$, and each $n_j>1$

The fixed set of T on $RP(n)$ is $RP(0)$ with trivial normal bundle, classified in $G_{n,n}$ and $RP(n-1)$ with the universal line bundle classified in $G_{1,n}$. The fixed data of $\Gamma(M)$ is obtained by adding a trivial line bundle to the fixed set of M, i.e. if $F^{n-j}\to G_{j,k}$, one is transformed to $G_{j+1,k+1}$ and by taking M with a trivial line bundle in $G_{1,1}$. For a product, the fixed data is classified via the Whitney sum maps $G_{j,k}\times G_{m,n}\to G_{j+m,k+n}$.

It is then immediate that for $\alpha=\Gamma R^iP(n_1)\times\cdots\times RP(n_k)\times M^m$, $\eta(\alpha)\leqq i+n_1+\cdots+n_k$.

The fixed component of least dimension in $\Gamma^i RP(n_1)\times\cdots\times RP(n_k)\times M^m$ is M^m with trivial normal bundle of dimension $j=i+n_1+\cdots+n_k$, and if $[M]\neq 0$ in $\mathfrak{N}_m$, this gives a nontrivial class in $\mathfrak{N}_m(G_{j,j})\subset\mathfrak{N}_m(BO_j)$, so $\eta(\alpha)\geqq i+n_1+\cdots+n_k$. Thus, one has Bix's theorem:

If $\alpha=\Gamma^i RP(n_1)\times RP(n_2)\times\cdots\times RP(n_k)\times M^m$ *with* $[M]\neq 0$ *in* $\mathfrak{N}_m$, *then* $\eta(\alpha)=i+n_1+\cdots+n_k$.

Reference

1. Michael C. Bix, *Z_2-equivariant immersions and embeddings up to cobordism*, Proc. Amer. Math. Soc. **44** (1973), 203–207.

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