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MAXWELL'S COEFFICIENTS ARE CONDITIONAL PROBABILITIES

REUBEN HERSH

ABSTRACT. The capacitance coefficients of electrostatics are represented as conditional probabilities associated with Brownian motion. It follows, as an immediate consequence, that these coefficients depend monotonically on their domains.

The "capacitance matrix" or "induction coefficients" of Maxwell are familiar in both electrostatics and the theory of conformal mapping. Let there be given a disjoint system of closed, bounded conductors $\Gamma_1, \cdots, \Gamma_n$ whose union Γ we regard as the boundary of a region $\mathscr{D}$ in the plane or in 3-space. Then the charges Q_i and the potentials U_j on these boundary components are related by a system of linear equations, $Q_i=\sum_{j=1}^n c_{ij}U_j$, whose coefficients c_{ij} depend only on the boundary surfaces Γ_i. Setting $U_i=\delta_{ij}$, we see that each coefficient c_{ij} gives the charge induced on Γ_i by a unit potential on Γ_j. In two dimensions, $\mathscr{D}$ is multiply connected, and the c_{ij} play a central role in the study of conformal mappings of multiply connected regions (see [5]).

It is physically obvious that $c_{ij}\geqq 0$ if $i\neq j$, and $c_{ii}\leqq 0$, and it is easily proved that $\langle c_{ij}\rangle$ is symmetric and positive semidefinite; the electrostatic energy of the system is $\frac{1}{2}\sum_{i,j=1}^n c_{ij}U_iU_j$. In three (or more) dimensions c_{ij} is nonsingular (see [2], [4]). In two dimensions, $\sum_{j=1}^n c_{ij}=0$, and zero is a simple eigenvalue of the matrix $\langle c_{ij}\rangle$, corresponding to the null vector $(1, 1, \cdots, 1)$ (see [5]).

In this note we give the probabilistic meaning of c_{ij}. This is so natural and simple, it is rather surprising it has not been noticed before now. In a sense we will make precise $-c_{ij}/c_{ii}$ is the conditional probability that a Brownian particle, which initially is near Γ_i, first meets the boundary at Γ_j. We use this interpretation to give a new proof that each c_{ij} is monotonic as a domain functional. That is, if all the conductors but the kth remain fixed, and the kth is enlarged, then c_{ik} increases, and $-c_{kk}$ increases, but

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c_{ij} and $-c_{ii}$ decrease (i, j and k all unequal to each other). From the probabilistic interpretation this property is immediate; to prove it analytically requires powerful tools (Hadamard's variational method).

In the special case $n=1$, we have only a single coefficient, which in the three-dimensional case is simply the capacity of Γ. This in itself is the subject of a well-known and elaborate theory.

Our analysis is based on two well-known facts about w_i, the ith harmonic measure on $\mathcal{D}$. By definition, w_i is the unique function which is harmonic in $\mathcal{D}$, is equal to 1 on Γ_i, and vanishes on Γ_j, $j \neq i$. In terms of electrostatics, w_i is the potential induced by raising the ith conductor to unit potential, and grounding the remaining $n-1$ conductors.

The first fact about w_i which we need is its connection with c_{ij}. If we let $k=1/2\pi$ in the two dimensional case and $k=1/4\pi$ in the three-dimensional case, and if Q is a variable point in Γ_i, then $c_{ij}=k\int_{\Gamma_i}(\partial w_j/\partial n)\,dQ$. This follows immediately from the elementary principle in electrostatics that the charge density on a single layer distribution is k times the normal derivative of the potential. Here $\partial w_j/\partial n$ is the normal derivative pointing from Γ_i into $\mathcal{D}$. If Γ_i is a "slit", then both normal directions point into $\mathcal{D}$, and the integration is over both sides of Γ_i.

The second fact we need about w_i is its probabilistic interpretation: $w_i(P)$ gives the probability that a Brownian particle, which initially is at P, is "absorbed" at Γ_i—i.e., it first meets Γ in Γ_i. (See [**1**] or [**3**].)

We want to compute the probability, which we will denote by p_{ij}, that a particle undergoing Brownian motion which manages to escape from the neighborhood of Γ_i is ultimately absorbed at Γ_j ($j \neq i$). The difficulty is that we are conditioning on an event with infinitesimal probability—the event that a particle close to Γ_i manages to be absorbed anywhere but at Γ_i. Nevertheless, we can give a meaningful definition in elementary terms, without resorting to sophisticated probabilistic tools. Let $S=S(i, \varepsilon)$ be the set of points in $\mathcal{D}$ whose distance from Γ_i is less than or equal to ε. We consider the sample space consisting of Brownian motions whose initial position is distributed at random, uniformly, over S. In this sample space, we let $A(i, j, \varepsilon)$ be the event that the path is absorbed at Γ_j—first meets the boundary Γ in its jth component. We let $B(i, \varepsilon)$ be the event that the path is *not* absorbed at Γ_i—that it meets Γ first at some Γ_j, $j \neq i$, or never meets any component of Γ. Now, for $i \neq j$, we simply *define* p_{ij} as the limit, as ε goes to zero, of the conditional probability of $A(i, j, \varepsilon)$ given $B(i, \varepsilon)$.

We first consider the case where the surfaces Γ_i are especially well-behaved. We call Γ_i "admissible" if it is smooth and has finite area and bounded normal curvature.

We let $\|S\|$ and $\|\Gamma_i\|$ denote the volume of S and the area of Γ_i. (In the two-dimensional case, of course, $\|S\|$ is an area and $\|\Gamma_i\|$ is a length.)

THEOREM. *If* Γ_i *is admissible*, $p_{ij}=-c_{ij}/c_{ii}$.

PROOF. We will prove that, for $i\neq j$,

$$c_{ij} = \lim_{\varepsilon\to 0} (2k\, \|\Gamma_i\|/\varepsilon)\text{prob } A(i, j, \varepsilon)$$

and

$$c_{ii} = -\lim_{\varepsilon\to 0} (2k\, \|\Gamma_i\|/\varepsilon)\text{prob } B(i, \varepsilon).$$

This will give the Theorem, because $A\subset B$, and therefore

$$\text{prob}(A/B) = \text{prob}(A \cap B)/\text{prob } B = \text{prob } A/\text{prob } B.$$

Now, in view of the probabilistic interpretation of w_j given above,

$$\text{prob } A = \int_S w_j(P)\, dP/\|S\| \quad \text{and} \quad \text{prob } B = \int_S (1 - w_i(P))\, dP/\|S\|.$$

Since Γ_i is smooth and has bounded normal curvature, there is at each point Q in Γ_i a unit normal vector $\boldsymbol{n}(Q)$ pointing into $\mathscr{D}$, and for ε sufficiently small each point P in $S(i, \varepsilon)$ has a unique representation $P=Q+r\boldsymbol{n}(Q)$, $0\leqq r\leqq\varepsilon$. The pair r, Q constitutes a normal-tangential coordinate system in S. In this coordinate system,

$$\int_S w_j(P)\, dP = \int_0^\varepsilon dr \int_{\Gamma_i} w_j(Q + r\boldsymbol{n}(Q))J(r, Q)\, dQ,$$

where the Jacobian $J(r, Q)$ satisfies $J(r, Q)=1+O(r)$ uniformly in Q. Using a finite Taylor expansion for w_j in powers of r, we have

$$\begin{aligned} w_j(P) &= w_j(Q) + r(\partial/\partial n)w_j(Q) + r^2\theta_j(P) \\ &= \delta_{ij} + r(\partial/\partial n)w_j(Q) + r^2\theta_j(P), \end{aligned}$$

where θ_j is a bounded function of P. Therefore

$$\begin{aligned} \int_S w_j(P)\, dP &= \int_0^\varepsilon dr \int_{\Gamma_i} w_j(Q + r\boldsymbol{n}(Q))J(r, Q)\, dQ \\ &= \delta_{ij}\, \|S\| + \frac{\varepsilon^2}{2}\int_{\Gamma_i} \frac{\partial}{\partial n} w_j(Q)\, dQ + R, \end{aligned}$$

where the error term R is $O(\varepsilon^2)$ uniformly in Q. Divide by $\|S\|$, observe that $\varepsilon/\|S\|\to 1/\|\Gamma_i\|$ as $\varepsilon\to 0$, and the conclusion follows.

Now suppose Γ_i has singularities, such as edges, corners, or cusps. If these singular points are a set of zero Hausdorff measure, the surface integrals c_{ij} are still well defined. If Γ_i is the uniform limit of a sequence of admissible surfaces $\Gamma_{i,n}$ which have uniformly bounded normal curvatures, and whose areas converge to the area of Γ_i, then a standard limiting argument again gives $p_{ij}=-c_{ij}/c_{ii}$. On the other hand, if Γ_i is everywhere or almost everywhere nondifferentiable, our theorem fails; indeed, in this case c_{ij} is not well defined, at least from the viewpoint of classical potential theory. Our proof suggests the possibility of defining a "generalized capacitance matrix" by

$$c_{ij} = \lim_{\varepsilon\to 0}(2k\|S\|/\varepsilon^2)\text{prob } A(i, j, \varepsilon)$$

and

$$c_{ii} = -\lim_{\varepsilon\to 0}(2k\|S\|/\varepsilon^2)\text{prob } B(i, \varepsilon).$$

COROLLARY 1. *Let $\mathscr{D}'_k\subset\mathscr{D}_k$, $\Gamma_j=\Gamma'_j$, $j\neq k$, Let c'_{ij} be the Maxwell coefficients associated with Γ'_j. Then, if i, j and k are all unequal,*

$$c'_{ik} \geqq c_{ik}, \quad -c'_{kk} \geqq -c_{kk}, \quad c'_{ij} \leqq c_{ij}, \quad -c'_{ii} \leqq -c_{ii}.$$

PROOF. If Γ_k is enlarged, while all other components of Γ are unchanged, then all the paths from $S(i, \varepsilon)$ which were formerly absorbed by Γ_k will still be absorbed by Γ'_k; moreover, some of the paths from $S(i, \varepsilon)$ which were absorbed by Γ_j will now be absorbed at Γ'_k. The conclusion follows.

COROLLARY 2. *The ratio $-c_{ij}/c_{ii}$ is a monotonic nondecreasing function of Γ_j and a monotonic nonincreasing function of Γ_k, $k\neq i$ and $k\neq j$.*

PROOF. The conditional probability of $A(i, j, \varepsilon)$ given $B(i, \varepsilon)$ is clearly increased by enlarging Γ_j and decreased by enlarging Γ_k. The conclusion follows by letting $\varepsilon\to 0$.

Brownian motion can also be used to prove some other properties of c_{ij} mentioned earlier. The symmetry of the matrix $\langle c_{ij}\rangle$ expresses the reversibility of Brownian motion. (That is, if T_1 is a collection of Brownian trajectories $x(t)$, $t_1\leqq t\leqq t_2$, satisfying $x(t_1)=P$, $x(t_2)=Q$, and if T_2 is the collection of reversed trajectories $\{y(t)\in T_2 \text{ iff } y(t)\equiv x(t_1+t_2-t), x(t)\in T_1\}$ then the conditional probability of the collection T_1, given $x(t_1)=P$, $x(t_2)=Q$, equals the conditional probability of T_2, given $x(t_1)=Q$, $x(t_2)=P$.)

The fact that $\sum_{j=1}^{n} c_{ij}=0$ in two dimensions, but not in three, is an expression of the fact that Brownian motion is recurrent in two dimensions and transient in three dimensions. That is, in three dimensions, but not in two, there is a positive probability that the path escapes to infinity without being absorbed at any Γ_i. In terms of p_{ij}, this means $\sum_j p_{ij}=1$ in two dimensions, but if $\mathscr{D}$ is unbounded, $\sum_j p_{ij}<1$ in three dimensions. If we regard infinity as an ideal $(n+1)$st component of Γ in the three-dimensional case, we would have $\sum_{j=1;j\neq i}^{n+1} p_{ij}=1$ and $\sum_{j=1}^{n+1} c_{ij}=0$.

If one of the components, say Γ_1, is a closed surface (or closed curve in the two-dimensional case) then, if Γ_i is in the interior of Γ_1 and Γ_j is in the exterior of Γ_1, we must have $c_{ij}=0$. This important elementary fact in electrostatics is now obvious from the continuity of the paths of the Brownian particle.

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Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87106