ON REGIONS OF \( \alpha \)-CONVEXITY FOR STARLIKE FUNCTIONS

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Abstract. In this note an alternative proof to a result of Mocanu and Reade [see Notices Amer. Math. Soc. 20 (1973), A-107] on \( \alpha \)-convexity region for starlike functions has been obtained.

1. Let \( f(z) = z + a_2 z^2 + \cdots \) be a regular function in \( |z| < 1 \) which satisfies the condition

\[
\Re \left( (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right) > 0
\]

for \( -\infty \leq \alpha \leq \infty \). Then \( f \) is said to be an \( \alpha \)-convex function. We denote the class of all such functions by \( S_\alpha \). If \( \alpha = 0 \), then \( f \) is starlike with respect to the origin. It is well known that \( f \in S_\alpha \) is univalent ([1], [2]). In this note an alternative proof of the following theorem of Mocanu and Reade [3] is obtained by a powerful classical method.

Theorem. If \( f(z) = z + a_2 z^2 + \cdots \) is univalent and starlike in the unit disc \( D \{ |z| < 1 \} \), then \( f(z) \) is \( \alpha \)-convex for \( |z| < r_0 \) where

\[
r_0 = (1 + \alpha) - ((1 + \alpha)^2 - 1)^{1/2} \quad \text{if} \quad \alpha \geq 0,
\]

\[
(1.2) \quad r_0 = \frac{(2 - (-\alpha)^{1/2})}{(2 + (-\alpha)^{1/2})}^{1/2} \quad \text{if} \quad -3 < \alpha \leq 0,
\]

\[
= -(1 + \alpha) - ((1 + \alpha)^2 - 1)^{1/2} \quad \text{if} \quad -3 \geq \alpha.
\]

2. Proof. Let \( w(z) \) be a regular function in \( D \) such that

\[
(2.1) \quad \frac{zf'(z)}{f(z)} = \frac{1 + w(z)}{1 - w(z)} \equiv p(z).
\]

Using a result of Schild [4], we obtain

\[
(2.2) \quad |p(z) - a| \leq d
\]

where

\[
(2.3) \quad a = \frac{1 + r^2}{1 - r^2}, \quad d = \frac{2r}{1 - r^2}.
\]

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Further, if we write,

\[ A(z) = \text{Re}\{p(z) - 1/p(z)\} \]

and

\[ B(z) = \frac{r^2 |p(z) + 1|^2 - |p(z) - 1|^2}{(1 - r^2) |p(z)|} , \]

then we obtain

\[ \frac{A(z) + B(z)}{4} \geq \text{Re}\left\{ \frac{zw'(z)}{1 - w^2(z)} \right\} \geq \frac{A(z) - B(z)}{4} . \]

In fact, (2.6) is a simple consequence of the following well-known inequality

\[ |zw'(z) - w(z)| \leq \frac{r^2 - |w(z)|^2}{1 - r^2} . \]

Now by differentiating (2.1) logarithmically, we get

\[ \text{Re}\left\{ \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) + (1 - \alpha) \frac{zf'(z)}{f(z)} \right\} = \text{Re}\left\{ \frac{1 + w(z)}{1 - w(z)} + \frac{2\alpha zw'(z)}{1 - w^2(z)} \right\} \]

\[ \geq \text{Re}\{p(z)\} + \alpha(A(z) - B(z))/2 \quad \text{if} \quad \alpha \geq 0 , \]

\[ \geq \text{Re}\{p(z)\} + \alpha(A(z) + B(z))/2 \quad \text{if} \quad \alpha < 0 . \]

First, let \( p(z) = a + u + iv, \ R_1^2 = (a + u^2 + v^2 \) and \( \alpha \geq 0 \). Then, we have

\[ \text{Re}\{p(z) + \alpha(A(z) - B(z))/2\} \]

\[ = \frac{1}{2} \left[ (a + u)^2 + \alpha - \frac{\alpha}{R_1^2} \right] - \alpha \left( \frac{d^2 - u^2 - v^2}{R_1^2} \right) \]

\[ \equiv S(u, v) . \]

From (2.9) we find

\[ \frac{\partial S(u, v)}{\partial v} = K(u, v)\left(\sqrt{v/R_1^2}\right) \quad \text{and} \quad K(u, v) > 0 . \]

Hence

\[ \min_v S(u, v) = S(u, 0) \equiv S(R) ; \quad R = a + u . \]

Further,

\[ S(R) = \frac{1}{2} \left[ (2 + \alpha)R - \frac{\alpha}{R} \right] - \alpha \left( \frac{d^2 - u^2 - v^2}{R} \right) \].

It is easy to see that \( S(R) \) is an increasing function of \( R \) and therefore

\[ \min_R S(R) = S(a - d) = \frac{1}{2} [(2 + \alpha)R - \alpha/R] \]

\[ = (1 - 2(1 + \alpha)R + r^2)/(1 - r^2) . \]
Now, let \( \alpha < 0 \). Write \( p(z) = Re^{i\theta} \) in the right-hand side of (2.8) and get

\[
2 \Re \{p(z) + \frac{1}{2} \alpha (A(z) + B(z))\} = \{(2 + \alpha) R - \alpha / R \} \cos \theta + \alpha \{-R - 1/R + 2a \cos \theta\} = J(R, \theta) \text{ (say)}.
\]

But

\[
\frac{\partial J(R, \theta)}{\partial \theta} = \{-[(2 + \alpha) R + \alpha (2a - 1/R)] \sin \theta = H(R, \alpha) \sin \theta
\]

where

\[
H(R, \alpha) = (\beta - 2) R + \beta (2a - 1/R), \quad \beta = -\alpha > 0.
\]

Since \( a - d \leq R \leq a + d \), we find

\[
(2a - 1/R) \geq 2a - 1/(a - d) = (1 - r)/(1 + r) > 0.
\]

Clearly, if \( \beta \geq 2 \) then \( H(R, \alpha) > 0 \). Hence

\[
\min_{\theta} J(R, \theta) = J(R) = 2(R - \alpha / R) + 2a \alpha.
\]

Further, if \( r_0' \) denotes the smallest positive root of the equation \( (\beta - 1) - 2(1 + \beta)r + (\beta - 1)r^2 = 0; \beta = -\alpha \geq 3; \) then

\[
\frac{\partial J(R)}{\partial R} = 2(\alpha + R^2)/R^2
\]

and

\[
R^2 + \alpha \leq \frac{(1 + \alpha) + 2(1 - \alpha)r + (1 + \alpha)r^2}{(1 - r^2)} \leq 0.
\]

Thus,

\[
\min_{R} J(R) = J(a + d)
\]

(2.18)

\[
= \frac{2[1 + 2(1 + \alpha)r + r^2]}{(1 - r^2)} \text{ if } r \leq r_0'.
\]

If \( r \geq r_0' \), then absolute minimum is obtained for \( R = (-\alpha)^{1/2} \) and \( 2 < -\alpha < 3 \). Thus

\[
\min_{R} J(R) = \frac{2}{R} \left[ -2\alpha + a\alpha (-\alpha)^{1/2} \right] = \frac{(-2\alpha)}{R} \left[ 2 - a(-\alpha)^{1/2} \right]
\]

(2.19)

\[
= \left( \frac{-2\alpha}{R} \right) \left[ \frac{(2 - (-\alpha)^{1/2}) - r^2(2 + (-\alpha)^{1/2})}{(1 - r^2)} \right].
\]

Again, let \(-2 < \alpha < -1\). Then, from (2.15) we obtain \( \partial H(R, \alpha) / \partial R = \beta - 2 + \beta / R^2 \), \( \beta = -\alpha \) and \( 1 < \beta < 2 \). This implies that \( H(R, \alpha) \) is an increasing
function of \( R \) whenever \( a - d \leq R \leq \beta^{1/2} \) and therefore

\[
\min_{R, \theta} J(R, \theta) = J(\beta^{1/2}, 0)
\]
as

\[
H(a - d, \alpha) = \frac{2(\beta - 1) + 4r + 2(\beta - 1)r^2}{(1 - r^2)} > 0.
\]

This is the same as (2.19). Finally, let \( 0 < \beta < 1 \). First, we restrict the values of \( r \) in the interval

\[
1 \leq \frac{3\beta^{1/2} - \beta \beta^{1/2} - 2\beta}{3\beta^{1/2} - \beta \beta^{1/2} + 2\beta} < r^2 < \frac{2 - \beta^{1/2}}{2 + \beta^{1/2}}.
\]

In this case, we find that the absolute relative minimum of \( J(R, \theta) \) again is obtained by (2.19). Hence, if \( r \) satisfies the inequalities (2.20) then \( f \in S_\alpha \) whenever \(-1 < \alpha < 0\). But at the origin the value of the expression on the left-hand side of (1.1) is one and is a harmonic function. But \( f \in S_\alpha \) in \(|z| \leq r; r \) satisfies (2.20)\}. By argument and the maximum principle of analytic functions it follows that \( f \in S_\alpha \) in \(|z| < r_0\). This completes the proof of the theorem as (2.12), (2.18) and (2.19) imply (1.2).

REFERENCES


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