

ON REGIONS OF α -CONVEXITY
 FOR STARLIKE FUNCTIONS

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ABSTRACT. In this note an alternative proof to a result of Mocanu and Reade [see Notices Amer. Math. Soc. 20 (1973), A-107] on α -convexity region for starlike functions has been obtained.

1. Let $f(z) = z + a_2z^2 + \dots$ be a regular function in $|z| < 1$ which satisfies the condition

$$(1.1) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0$$

for $-\infty \leq \alpha \leq \infty$. Then f is said to be an α -convex function. We denote the class of all such functions by S_α . If $\alpha = 0$, then f is starlike with respect to the origin. It is well known that $f \in S_\alpha$ is univalent ([1], [2]). In this note an alternative proof of the following theorem of Mocanu and Reade [3] is obtained by a powerful classical method.

THEOREM. If $f(z) = z + a_2z^2 + \dots$ is univalent and starlike in the unit disc $D\{|z| < 1\}$, then $f(z)$ is α -convex for $|z| < r_0$ where

$$(1.2) \quad \begin{aligned} r_0 &= (1 + \alpha) - ((1 + \alpha)^2 - 1)^{1/2} && \text{if } \alpha \geq 0, \\ &= \{(2 - (-\alpha)^{1/2}) / (2 + (-\alpha)^{1/2})\}^{1/2} && \text{if } -3 < \alpha \leq 0, \\ &= -(1 + \alpha) - ((1 + \alpha)^2 - 1)^{1/2} && \text{if } -3 \geq \alpha. \end{aligned}$$

2. PROOF. Let $w(z)$ be a regular function in D such that

$$(2.1) \quad \frac{zf'(z)}{f(z)} = \frac{1 + w(z)}{1 - w(z)} \equiv p(z).$$

Using a result of Schild [4], we obtain

$$(2.2) \quad |p(z) - a| \leq d$$

where

$$(2.3) \quad a = \frac{1 + r^2}{1 - r^2}, \quad d = \frac{2r}{1 - r^2}.$$

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Further, if we write,

$$(2.4) \quad A(z) = \operatorname{Re}\{p(z) - 1/p(z)\}$$

and

$$(2.5) \quad B(z) = \frac{r^2 |p(z) + 1|^2 - |p(z) - 1|^2}{(1 - r^2) |p(z)|},$$

then we obtain

$$(2.6) \quad \frac{A(z) + B(z)}{4} \geq \operatorname{Re}\left\{\frac{zw'(z)}{1 - w^2(z)}\right\} \geq \frac{A(z) - B(z)}{4}.$$

In fact, (2.6) is a simple consequence of the following well-known inequality

$$(2.7) \quad |zw'(z) - w(z)| \leq \frac{r^2 - |w(z)|^2}{1 - r^2}.$$

Now by differentiating (2.1) logarithmically, we get

$$(2.8) \quad \begin{aligned} \operatorname{Re}\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1 - \alpha)\frac{zf'(z)}{f(z)}\right\} &= \operatorname{Re}\left\{\frac{1 + w(z)}{1 - w(z)} + \frac{2\alpha zw'(z)}{1 - w^2(z)}\right\} \\ &\geq \operatorname{Re}\{p(z)\} + \alpha(A(z) - B(z))/2 \quad \text{if } \alpha \geq 0, \\ &\geq \operatorname{Re}\{p(z)\} + \alpha(A(z) + B(z))/2 \quad \text{if } \alpha < 0. \end{aligned}$$

First, let $p(z) = a + u + iv$, $R_1^2 = (a + u)^2 + v^2$ and $\alpha \geq 0$. Then, we have

$$(2.9) \quad \begin{aligned} &\operatorname{Re}\{p(z) + \alpha(A(z) - B(z))/2\} \\ &= \frac{1}{2}\left[(a + u)\left\{2 + \alpha - \frac{\alpha}{R_1^2}\right\} - \alpha\left(\frac{d^2 - u^2 - v^2}{R_1}\right)\right] \\ &\equiv S(u, v). \end{aligned}$$

From (2.9) we find

$$(2.10) \quad \frac{\partial S(u, v)}{\partial v} = K(u, v)(v/R_1^4) \quad \text{and} \quad K(u, v) > 0.$$

Hence

$$(2.11) \quad \min_v S(u, v) = S(u, 0) \equiv S(R); \quad R = a + u.$$

Further,

$$S(R) = \frac{1}{2}\left[\left\{(2 + \alpha)R - \frac{\alpha}{R}\right\} - \alpha\left(\frac{d^2 - u^2}{R}\right)\right].$$

It is easy to see that $S(R)$ is an increasing function of R and therefore

$$(2.12) \quad \begin{aligned} \min_R S(R) &= S(a - d) = \frac{1}{2}[(2 + \alpha)R - \alpha/R] \\ &= (1 - 2(1 + \alpha)r + r^2)/(1 - r^2). \end{aligned}$$

Now, let $\alpha < 0$. Write $p(z) = Re^{i\theta}$ in the right-hand side of (2.8) and get

$$(2.13) \quad \begin{aligned} & 2 \operatorname{Re}\{p(z) + \frac{1}{2}\alpha(A(z) + B(z))\} \\ &= \{(2 + \alpha)R - \alpha/R\}\cos \theta + \alpha\{-R - 1/R + 2a \cos \theta\} \\ &\equiv J(R, \theta) \quad (\text{say}). \end{aligned}$$

But

$$(2.14) \quad \frac{\partial J(R, \theta)}{\partial \theta} = -[(2 + \alpha)R + \alpha(2a - 1/R)]\sin \theta = H(R, \alpha)\sin \theta$$

where

$$(2.15) \quad H(R, \alpha) = (\beta - 2)R + \beta(2a - 1/R), \quad \beta = -\alpha > 0.$$

Since $(a-d) \leq R \leq (a+d)$, we find

$$(2a - 1/R) \geq 2a - 1/(a-d) = (1-r)/(1+r) > 0.$$

Clearly, if $\beta \geq 2$ then $H(R, \alpha) > 0$. Hence

$$\min_{\theta} J(R, \theta) \equiv J(R) = 2(R - \alpha/R) + 2a\alpha.$$

Further, if r'_0 denotes the smallest positive root of the equation $(\beta-1) - 2(1+\beta)r + (\beta-1)r^2 = 0$; $\beta = -\alpha \geq 3$; then

$$(2.16) \quad \partial J(R)/\partial R = 2(\alpha + R^2)/R^2$$

and

$$(2.17) \quad R^2 + \alpha \leq \frac{(1 + \alpha) + 2(1 - \alpha)r + (1 + \alpha)r^2}{(1 - r^2)} \leq 0.$$

Thus,

$$(2.18) \quad \begin{aligned} \min_R J(R) &= J(a+d) \\ &= \frac{2[1 + 2(1 + \alpha)r + r^2]}{(1 - r^2)} \quad \text{if } r \leq r'_0. \end{aligned}$$

If $r \not\leq r'_0$, then absolute minimum is obtained for $R = (-\alpha)^{1/2}$ and $2 < -\alpha < 3$. Thus

$$(2.19) \quad \begin{aligned} \min_R J(R) &= \frac{2}{R} [-2\alpha + a\alpha(-\alpha)^{1/2}] = \frac{(-2\alpha)}{R} [2 - a(-\alpha)^{1/2}] \\ &= \left(\frac{-2\alpha}{R}\right) \left[\frac{(2 - (-\alpha)^{1/2}) - r^2(2 + (-\alpha)^{1/2})}{(1 - r^2)}\right]. \end{aligned}$$

Again, let $-2 < \alpha < -1$. Then, from (2.15) we obtain $\partial H(R, \alpha)/\partial R = \beta - 2 + \beta/R^2$, $\beta = -\alpha$ and $1 < \beta < 2$. This implies that $H(R, \alpha)$ is an increasing

function of R whenever $a-d \leq R \leq \beta^{1/2}$ and therefore

$$\min_{R, \theta} J(R, \theta) = J(\beta^{1/2}, 0)$$

as

$$H(a-d, \alpha) = \frac{2(\beta-1) + 4r + 2(\beta-1)r^2}{(1-r^2)} > 0.$$

This is the same as (2.19). Finally, let $0 < \beta < 1$. First, we restrict the values of r in the interval

$$(2.20) \quad \frac{3\beta^{1/2} - \beta\beta^{1/2} - 2\beta}{3\beta^{1/2} - \beta\beta^{1/2} + 2\beta} < r^2 < \frac{2 - \beta^{1/2}}{2 + \beta^{1/2}}.$$

In this case, we find that the absolute relative minimum of $J(R, \theta)$ again is obtained by (2.19). Hence, if r satisfies the inequalities (2.20) then $f \in S_\alpha$ whenever $-1 < \alpha < 0$. But at the origin the value of the expression on the left-hand side of (1.1) is one and is a harmonic function. But $f \in S_\alpha$ in $\{|z| \leq r; r \text{ satisfies (2.20)}\}$. By argument and the maximum principle of analytic functions it follows that $f \in S_\alpha$ in $|z| < r_0$. This completes the proof of the theorem as (2.12), (2.18) and (2.19) imply (1.2).

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