PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

ON REGIONS OF α-CONVEXITY FOR STARLIKE FUNCTIONS

S. K. BAJPAI

ABSTRACT. In this note an alternative proof to a result of Mocanu and Reade [see Notices Amer. Math. Soc. **20** (1973), A-107] on α-convexity region for starlike functions has been obtained.

1. Let $f(z)=z+a_2z^2+\cdots$ be a regular function in $|z|<1$ which satisfies the condition

$$\operatorname{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>0 \tag{1.1}$$

for $-\infty\leqq\alpha\leqq\infty$. Then f is said to be an α-convex function. We denote the class of all such functions by S_α. If $\alpha=0$, then f is starlike with respect to the origin. It is well known that $f\in S_\alpha$ is univalent ([**1**], [**2**]). In this note an alternative proof of the following theorem of Mocanu and Reade [**3**] is obtained by a powerful classical method.

THEOREM. *If $f(z)=z+a_2z^2+\cdots$ is univalent and starlike in the unit disc $D\{|z|<1\}$, then $f(z)$ is α-convex for $|z|<r_0$ where*

$$\begin{aligned} r_0 &= (1+\alpha)-((1+\alpha)^2-1)^{1/2} && \text{if } \alpha\geqq 0,\\ &= \{(2-(-\alpha)^{1/2})/(2+(-\alpha)^{1/2})\}^{1/2} && \text{if } -3<\alpha\leqq 0,\\ &= -(1+\alpha)-((1+\alpha)^2-1)^{1/2} && \text{if } -3\geqq\alpha. \end{aligned} \tag{1.2}$$

2. PROOF. Let $w(z)$ be a regular function in D such that

$$\frac{zf'(z)}{f(z)}=\frac{1+w(z)}{1-w(z)}\equiv p(z). \tag{2.1}$$

Using a result of Schild [**4**], we obtain

$$|p(z)-a|\leqq d \tag{2.2}$$

where

$$a=\frac{1+r^2}{1-r^2},\qquad d=\frac{2r}{1-r^2}. \tag{2.3}$$

Received by the editors March 6, 1973.

AMS (MOS) subject classifications (1970). Primary 30A32.

Key words and phrases. Starlike, α-starlike, α-convex functions.

Further, if we write,

$$A(z) = \operatorname{Re}\{p(z) - 1/p(z)\} \tag{2.4}$$

and

$$B(z) = \frac{r^2\,|p(z)+1|^2 - |p(z)-1|^2}{(1-r^2)\,|p(z)|}, \tag{2.5}$$

then we obtain

$$\frac{A(z)+B(z)}{4} \geqq \operatorname{Re}\left\{\frac{zw'(z)}{1-w^2(z)}\right\} \geqq \frac{A(z)-B(z)}{4}. \tag{2.6}$$

In fact, (2.6) is a simple consequence of the following well-known inequality

$$|zw'(z) - w(z)| \leqq \frac{r^2 - |w(z)|^2}{1-r^2}. \tag{2.7}$$

Now by differentiating (2.1) logarithmically, we get

$$\begin{aligned} \operatorname{Re}\left\{\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)}\right\} &= \operatorname{Re}\left\{\frac{1+w(z)}{1-w(z)} + \frac{2\alpha zw'(z)}{1-w^2(z)}\right\} \\ &\geqq \operatorname{Re}\{p(z)\} + \alpha(A(z)-B(z))/2 \quad \text{if } \alpha \geqq 0, \\ &\geqq \operatorname{Re}\{p(z)\} + \alpha(A(z)+B(z))/2 \quad \text{if } \alpha < 0. \end{aligned} \tag{2.8}$$

First, let $p(z)=a+u+iv$, $R_1^2=(a+u)^2+v^2$ and $\alpha \geqq 0$. Then, we have

$$\begin{aligned} &\operatorname{Re}\{p(z) + \alpha(A(z)-B(z))/2\} \\ &\quad = \frac{1}{2}\left[(a+u)\left\{2+\alpha-\frac{\alpha}{R_1^2}\right\} - \alpha\left(\frac{d^2-u^2-v^2}{R_1}\right)\right] \\ &\quad \equiv S(u,v). \end{aligned} \tag{2.9}$$

From (2.9) we find

$$\frac{\partial S(u,v)}{\partial v} = K(u,v)(v/R_1^4) \quad \text{and} \quad K(u,v) > 0. \tag{2.10}$$

Hence

$$\min_v S(u,v) = S(u,0) \equiv S(R); \qquad R = a+u. \tag{2.11}$$

Further,

$$S(R) = \frac{1}{2}\left[\left\{(2+\alpha)R - \frac{\alpha}{R}\right\} - \alpha\left(\frac{d^2-u^2}{R}\right)\right].$$

It is easy to see that $S(R)$ is an increasing function of R and therefore

$$\begin{aligned} \min_R S(R) &= S(a-d) = \tfrac{1}{2}[(2+\alpha)R - \alpha/R] \\ &= (1-2(1+\alpha)r + r^2)/(1-r^2). \end{aligned} \tag{2.12}$$

Now, let $\alpha<0$. Write $p(z)=Re^{i\theta}$ in the right-hand side of (2.8) and get

$$\begin{aligned}(2.13)\qquad & 2\,\mathrm{Re}\{p(z)+\tfrac{1}{2}\alpha(A(z)+B(z))\}\\ &\quad =\{(2+\alpha)R-\alpha/R\}\cos\theta+\alpha\{-R-1/R+2a\cos\theta\}\\ &\quad \equiv J(R,\theta)\quad \text{(say)}.\end{aligned}$$

But

$$(2.14)\qquad \frac{\partial J(R,\theta)}{\partial\theta}=-[(2+\alpha)R+\alpha(2a-1/R)]\sin\theta=H(R,\alpha)\sin\theta$$

where

$$(2.15)\qquad H(R,\alpha)=(\beta-2)R+\beta(2a-1/R),\quad \beta=-\alpha>0.$$

Since $(a-d)\leqq R\leqq(a+d)$, we find

$$(2a-1/R)\geqq 2a-1/(a-d)=(1-r)/(1+r)>0.$$

Clearly, if $\beta\geqq 2$ then $H(R,\alpha)>0$. Hence

$$\min_{\theta} J(R,\theta)\equiv J(R)=2(R-\alpha/R)+2a\alpha.$$

Further, if r_0' denotes the smallest positive root of the equation $(\beta-1)-2(1+\beta)r+(\beta-1)r^2=0$; $\beta=-\alpha\geqq 3$; then

$$(2.16)\qquad \partial J(R)/\partial R=2(\alpha+R^2)/R^2$$

and

$$(2.17)\qquad R^2+\alpha\leqq\frac{(1+\alpha)+2(1-\alpha)r+(1+\alpha)r^2}{(1-r^2)}\leqq 0.$$

Thus,

$$\begin{aligned}(2.18)\qquad \min_{R} J(R)&=J(a+d)\\ &=\frac{2[1+2(1+\alpha)r+r^2]}{(1-r^2)}\quad \text{if } r\leqq r_0'.\end{aligned}$$

If $r\nleqq r_0'$, then absolute minimum is obtained for $R=(-\alpha)^{1/2}$ and $2<-\alpha<3$. Thus

$$\begin{aligned}(2.19)\qquad \min_{R} J(R)&=\frac{2}{R}[-2\alpha+a\alpha(-\alpha)^{1/2}]=\frac{(-2\alpha)}{R}[2-a(-\alpha)^{1/2}]\\ &=\left(\frac{-2\alpha}{R}\right)\left[\frac{(2-(-\alpha)^{1/2})-r^2(2+(-\alpha)^{1/2})}{(1-r^2)}\right].\end{aligned}$$

Again, let $-2<\alpha<-1$. Then, from (2.15) we obtain $\partial H(R,\alpha)/\partial R=\beta-2+\beta/R^2$, $\beta=-\alpha$ and $1<\beta<2$. This implies that $H(R,\alpha)$ is an increasing function of R whenever $a-d \leqq R \leqq \beta^{1/2}$ and therefore

$$\min_{R,\theta} J(R, \theta) = J(\beta^{1/2}, 0)$$

as

$$H(a-d, \alpha) = \frac{2(\beta-1) + 4r + 2(\beta-1)r^2}{(1-r^2)} > 0.$$

This is the same as (2.19). Finally, let $0<\beta<1$. First, we restrict the values of r in the interval

$$\frac{3\beta^{1/2} - \beta\beta^{1/2} - 2\beta}{3\beta^{1/2} - \beta\beta^{1/2} + 2\beta} < r^2 < \frac{2-\beta^{1/2}}{2+\beta^{1/2}}. \tag{2.20}$$

In this case, we find that the absolute relative minimum of $J(R, \theta)$ again is obtained by (2.19). Hence, if r satisfies the inequalities (2.20) then $f \in S_\alpha$ whenever $-1<\alpha<0$. But at the origin the value of the expression on the left-hand side of (1.1) is one and is a harmonic function. But $f \in S_\alpha$ in $\{|z| \leqq r;\ r$ satisfies (2.20)$\}$. By argument and the maximum principle of analytic functions it follows that $f \in S_\alpha$ in $|z|<r_0$. This completes the proof of the theorem as (2.12), (2.18) and (2.19) imply (1.2).

References

1. S. K. Bajpai and E. M. Silvia, *On certain classes of analytic functions*, Notices Amer. Math. Soc. **20** (1973), A-107. Abstract #701-30-17.
2. S. S. Miller, P. T. Mocanu and M. O. Reade, *All α-convex functions are starlike*, Rev. Roumaine Math. Pures Appl. **17** (1972), 1305–1307.
3. P. T. Mocanu and M. O. Reade, *The radius of α-convexity for the class of starlike univalent functions, α-real*, Notices Amer. Math. Soc. **20** (1973). A-521. Abstract #706-30-3.
4. Albert Schild, *On starlike functions of order α*, Amer. J. Math. **87** (1965), 65–70. MR **30** #4929.

Department of Mathematics, Clark University, Worcester, Massachusetts 01610