ON REGIONS OF $\alpha$-CONVEXITY
FOR STARLIKE FUNCTIONS

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Abstract. In this note an alternative proof to a result of
Mocanu and Reade [see Notices Amer. Math. Soc. 20 (1973),
A-107] on $\alpha$-convexity region for starlike functions has been
obtained.

1. Let $f(z) = z + a_2 z^2 + \cdots$ be a regular function in $|z| < 1$ which satisfies
the condition

$$\Re \left( (1 - \alpha) \frac{zf''(z)}{f(z)} + \alpha \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right) > 0$$

for $-\infty \leq \alpha \leq \infty$. Then $f$ is said to be an $\alpha$-convex function. We denote
the class of all such functions by $S_{\alpha}$. If $\alpha = 0$, then $f$ is starlike with respect
to the origin. It is well known that $f \in S_{\alpha}$ is univalent ([1], [2]). In this
note an alternative proof of the following theorem of Mocanu and Reade
[3] is obtained by a powerful classical method.

Theorem. If $f(z) = z + a_2 z^2 + \cdots$ is univalent and starlike in the unit
disc $D(|z| < 1)$, then $f(z)$ is $\alpha$-convex for $|z| < r_0$ where

$$r_0 = (1 + \alpha) - ((1 + \alpha)^2 - 1)^{1/2} \quad \text{if } \alpha \geq 0,$$

$$= \{2 - (-\alpha)^{1/2}/(2 + (-\alpha)^{1/2})\}^{1/2} \quad \text{if } -3 < \alpha \leq 0,$$

$$= -(1 + \alpha) - ((1 + \alpha)^2 - 1)^{1/2} \quad \text{if } -3 \geq \alpha.$$

2. Proof. Let $w(z)$ be a regular function in $D$ such that

$$\frac{zf''(z)}{f(z)} = \frac{1 + w(z)}{1 - w(z)} \equiv p(z).$$

Using a result of Schild [4], we obtain

$$|p(z) - a| \leq d$$

where

$$a = \frac{1 + r^2}{1 - r^2}, \quad d = \frac{2r}{1 - r^2}.$$

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Further, if we write,

\[ A(z) = \text{Re}\{p(z) - 1/p(z)\} \]

and

\[ B(z) = \frac{r^2 |p(z) + 1|^2 - |p(z) - 1|^2}{(1 - r^2) |p(z)|} , \]

then we obtain

\[ \frac{A(z) + B(z)}{4} \geq \text{Re}\left( \frac{zw'(z)}{1 - w^2(z)} \right) \geq \frac{A(z) - B(z)}{4} . \]

In fact, (2.6) is a simple consequence of the following well-known inequality

\[ |zw'(z) - w(z)| \leq \frac{r^2 - |w(z)|^2}{1 - r^2} . \]

Now by differentiating (2.1) logarithmically, we get

\[ \text{Re}\left( \frac{1}{1 - w(z)} - \frac{2 \alpha z w'(z)}{1 - w^2(z)} \right) \geq \frac{1}{2} \text{Re}\{p(z) + \alpha(A(z) - B(z))/2 \} \text{ if } \alpha \geq 0 , \]

\[ \geq \text{Re}\{p(z)\} + \alpha(A(z) + B(z))/2 \text{ if } \alpha < 0 . \]

First, let \( p(z) = a + u + iv, R_1^2 = (a + u)^2 + v^2 \) and \( \alpha \geq 0 \). Then, we have

\[ \text{Re}\{p(z) + \alpha(A(z) - B(z))/2\} \]

\[ = \frac{1}{2}\left[ (a + u)^2 + \alpha - \frac{\alpha}{R_1^2} - \alpha \left( \frac{u^2 - v^2}{R_1^2} \right) \right] \]

\[ = S(u, v) . \]

From (2.9) we find

\[ \frac{\partial S(u, v)}{\partial v} = K(u, v)(v/R_1^4) \text{ and } K(u, v) > 0 . \]

Hence

\[ \min_v S(u, v) = S(u, 0) \equiv S(R); \quad R = a + u . \]

Further,

\[ S(R) = \frac{1}{2}\left[ (2 + \alpha)R - \frac{\alpha}{R} \right] - \alpha \left( \frac{R^2 - u^2}{R} \right) . \]

It is easy to see that \( S(R) \) is an increasing function of \( R \) and therefore

\[ \min_R S(R) = S(a - d) = \frac{1}{2}[(2 + \alpha)R - \alpha/R] \]

\[ = (1 - 2(1 + \alpha)r + r^2)/(1 - r^2) . \]
Now, let \( \alpha < 0 \). Write \( p(z) = Re^{i\theta} \) in the right-hand side of (2.8) and get

\[
2 \text{Re}\{p(z) + \frac{1}{2} \alpha (A(z) + B(z))\} = \{(2 + \alpha)R - \alpha/R\} \cos \theta + \alpha\{-R - 1/R + 2\alpha \cos \theta\}
\]

\( \equiv J(R, \theta) \) (say).

But

\[
\frac{\partial J(R, \theta)}{\partial \theta} = -\{(2 + \alpha)R + \alpha(2a - 1/R)\} \sin \theta = H(R, \alpha) \sin \theta
\]

where

\[
H(R, \alpha) = (\beta - 2)R + \beta(2a - 1/R), \quad \beta = -\alpha > 0.
\]

Since \((a - d) \leq R \leq (a + d)\), we find

\[
(2a - 1/R) \geq 2a - 1/(a - d) = (1 - r)/(1 + r) > 0.
\]

Clearly, if \( \beta \geq 2 \) then \( H(R, \alpha) > 0 \). Hence

\[
\min_{\theta} J(R, \theta) \equiv J(R) = 2(R - \alpha/R) + 2\alpha a.
\]

Further, if \( r' = 0 \) denotes the smallest positive root of the equation \((\beta - 1) - 2(1+\beta)r + (\beta-1)r^2 = 0; \beta = -\alpha \geq 3 \); then

\[
\frac{\partial J(R)}{\partial R} = 2(\alpha + R^2)/R^2
\]

and

\[
R^2 + \alpha \leq \frac{(1 + \alpha) + 2(1 - \alpha)r + (1 + \alpha)r^2}{(1 - r^2)} \leq 0.
\]

Thus,

\[
\min J(R) = J(a + d)
\]

\[
= \frac{2[1 + 2(1 + \alpha)r + r^2]}{(1 - r^2)} \quad \text{if} \quad r \leq r'_0.
\]

If \( r \geq r'_0 \), then absolute minimum is obtained for \( R = (-\alpha)^{1/2} \) and \( 2 < -\alpha < 3 \). Thus

\[
\min J(R) = \frac{2}{R} \left[-2\alpha + a\alpha(-\alpha)^{1/2}\right] = \frac{(-2\alpha)}{R} \left[2 - a(-\alpha)^{1/2}\right]
\]

\[
\left(\frac{-2\alpha}{R}\right) \left[\frac{(2 - (-\alpha)^{1/2}) - r^2(2 + (-\alpha)^{1/2})}{(1 - r^2)}\right].
\]

Again, let \(-2 < \alpha < -1\). Then, from (2.15) we obtain \( \partial H(R, \alpha)/\partial R = \beta - 2 + \beta/R^2, \beta = -\alpha \) and \( 1 < \beta < 2 \). This implies that \( H(R, \alpha) \) is an increasing
function of $R$ whenever $a - d \leq R \leq \beta^{1/2}$ and therefore

$$\min_{R, \theta} J(R, \theta) = J(\beta^{1/2}, 0)$$

as

$$H(a - d, \alpha) = \frac{2(\beta - 1) + 4r + 2(\beta - 1)r^2}{(1 - r^2)} > 0.$$ 

This is the same as (2.19). Finally, let $0 < \beta < 1$. First, we restrict the values of $r$ in the interval

$$1^{1/2} \beta^{1/2} - \beta^{1/2} + 2\beta^{1/2} + 2/3 < r < 2 - \beta^{1/2}.$$ (2.20)

In this case, we find that the absolute relative minimum of $J(R, \theta)$ again is obtained by (2.19). Hence, if $r$ satisfies the inequalities (2.20) then $f \in S_x$ whenever $-1 < \alpha < 0$. But at the origin the value of the expression on the left-hand side of (1.1) is one and is a harmonic function. But $f \in S_x$ in $\{|z| \leq r; r \text{ satisfies (2.20)}\}$. By argument and the maximum principle of analytic functions it follows that $f \in S_x$ in $|z| < r_0$. This completes the proof of the theorem as (2.12), (2.18) and (2.19) imply (1.2).

REFERENCES


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