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ASYMPTOTIC STABILITY FOR SOME CRITICAL AUTONOMOUS DIFFERENTIAL EQUATIONS

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ABSTRACT. Liapunov functions are constructed and used to prove stability theorems for critical autonomous systems in which the linear part of the right-hand side has a zero eigenvalue.

1. **Introduction.** As an application of his celebrated "second method", Liapunov [3] proved the following theorem dealing with the critical case for an autonomous system of ordinary differential equations in which the linear part of the right-hand side has a zero eigenvalue.

THEOREM 1.1 (LIAPUNOV). *Let x be an n-vector, y a scalar, and A an $n\times n$ stable matrix. Consider the system of $n+1$ equations*

$$\begin{aligned}\dot{x} &= Ax + c(x_ix_j) + c(x_iy) + c(y^{m+1}),\\ \dot{y} &= by^m + c(x_ix_jx_k) + c(x_ix_jy) + c(y^{m+1}),\end{aligned}\tag{1.1}$$

where $i, j, k=1, 2, \cdots, n$, $m\geqq 2$, $b\neq 0$, and $c(z)$ represents collections of terms (vectors or scalars) which have a common factor of z. If m is odd and $b<0$, then zero is uniformly asymptotically stable; otherwise, zero is unstable.

In this paper, we prove a stronger version of this theorem via Liapunov's method by using a function which is a modification of the one originally used by Liapunov; for the special case of two dimensions (when $n=1$), we obtain yet a more general result with a Liapunov function motivated by the following considerations. If $p_y(x, y)=q_x(x, y)$, then $p(x, y)\,dx+q(x, y)\,dy$ is an exact differential, and hence the function

$$W(x, y) = \int_{(0,0)}^{(x,y)} p(u, v)\,du + q(u, v)\,dv$$

is well defined because the line integral is independent of path. Moreover, curves defined by $W(x, y)=\alpha$, where α is a real parameter, are orthogonal to trajectories of

$$\dot{x} = p(x, y), \qquad \dot{y} = q(x, y).\tag{1.2}$$

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Thus, with suitable hypotheses on $p(x, y)$ and $q(x, y)$, $W(x, y)$ becomes a candidate for a Liapunov function. For the more general situation in which the appropriate partial derivatives of $p(x, y)$ and $q(x, y)$ are not necessarily equal, we may consider the "completed" differential form

$$\left[p(x, y) + \int_0^y (q_x(x, v) - p_y(x, v))\, dv\right] dx + q(x, y)\, dy,$$

which is always exact. If $q_x(x, y) - p_y(x, y)$ is small in some sense, we may hope that

$$\begin{aligned} V(x, y) &= \int_{(0,0)}^{(x,y)} \left[p(u, 0) + \int_0^v q_x(u, w)\, dw\right] du + q(u, v)\, dv \\ &= \int_0^x p(u, 0)\, du + \int_0^y q(x, v)\, dv \end{aligned}$$

defines a Liapunov function for (1.2). Other interesting constructions of Liapunov functions are discussed in LaSalle and Lefschetz [**2**] and Leighton [**4**].

2. **Preliminaries.** Let R^n be Euclidean n-space with norm $|\cdot|$; if $x \in R^n$, denote the transpose of x by x^T. Let $F: D \to R^n$, where D is some domain in R^n containing the origin, and assume that solutions of the autonomous differential equation

$$\dot{x} = F(x) \tag{2.1}$$

exist and are unique. Let $F(0)=0$ so that the zero function is a solution of (2.1). If $\phi(t, \tau, \xi)$ represents the solution of (2.1) such that $\phi(\tau, \tau, \xi) = \xi$, assume that $\phi(t, \tau, \xi)$ exists for $t \geqq \tau$ for all $\tau \geqq 0$ and sufficiently small $|\xi|$.

The various stability definitions of the zero solution of (2.1) can be found in LaSalle and Lefschetz [**2**], along with the basic ideas of Liapunov theory. In particular, the proofs of §3 rest on Theorems II and III of that book.

3. **Results.** The following simple lemma is well known and can be found in Hale [**1**].

LEMMA 3.1. *Suppose A is a real $n \times n$ matrix. The matrix equation $A^T B + BA = -C$ has a positive definite solution matrix B for every positive definite matrix C if and only if A is a stable matrix, that is, the real parts of all the eigenvalues of A are negative.*

THEOREM 3.2. *Let A be be an $n \times n$ stable matrix, and consider the system of $n+1$ equations*

$$\begin{aligned} \dot{x} &= Ax + c(x_i x_j) + c(x_i y) + c(y^{m+1}), \\ \dot{y} &= by^m + c(x_i x_j) + c(x_i y) + c(y^{m+1}), \end{aligned} \tag{3.1}$$

where $i, j=1, 2, \cdots, n$, $m \geqq 2$, *and* $b \neq 0$. *If* m *is odd and* $b<0$, *then the zero solution of* (3.1) *is uniformly asymptotically stable*; *otherwise*, *zero is unstable.*

PROOF. Let C be a positive definite matrix with all positive components such that $x^T C x \geqq 2|x|^2$; let B be a positive definite matrix such that $A^T B + BA = -C$. Define

$$\begin{aligned}
& V(x, y) = x^T B x - b y^{m+1}/(m+1) \\
\Rightarrow \quad & \dot{V}(x, y) = \dot{x}^T B x + x^T B \dot{x} - b y^m \dot{y} \\
& \qquad = (x^T A^T + c^T(x_i x_j) + c^T(x_i y) + c^T(y^{m+1}))Bx \\
& \qquad\quad + x^T B(Ax + c(x_i x_j) + c(x_i y) + c(y^{m+1})) \\
& \qquad\quad - b y^m (b y^m + c(x_i x_j) + c(x_i y) + c(y^{m+1})) \\
\Rightarrow \quad & -\dot{V}(x, y) = x^T C x + b^2 y^{2m} + c(x_i x_j x_k) \\
& \qquad\quad + c(x_i x_j y) + c(x_i y^{m+1}) + c(y^{2m+1}).
\end{aligned}$$

Since C has positive components, the quadratic form $x^T C x$ dominates the terms $c(x_i x_j x_k)$ and $c(x_i x_j y)$ for small $|x|$ and $|y|$ so that

$$-\dot{V}(x, y) \geqq x^T \hat{C} x + b^2 y^{2m} + c(x_i y^{m+1}) + c(y^{2m+1}),$$

where $\hat{C}$ is a positive definite matrix with positive components such that $x^T \hat{C} x \geqq |x|^2$. Hence, for small $|x|$ and $|y|$,

$$\begin{aligned}
-\dot{V}(x, y) &\geqq |x|^2 + \theta b^2 y^{2m} - 2M\,|x|\,|y|^{m+1} \\
&= (|x| - M\,|y|^{m+1})^2 + (\theta b^2 - M^2 y^2) y^{2m},
\end{aligned}$$

where $M>0$ and $\theta = 1^-$. Thus, $-\dot{V}(x, y)$ is positive definite in a neighborhood of the origin. Finally, if $b<0$ and m is odd, $V(x, y)$ is positive definite; otherwise, $-by^{m+1}$ is negative for certain arbitrarily small values of y, and therefore, $V(x, y)$ is negative arbitrarily near the origin along the y-axis. □

The next theorem is concerned with the two-dimensional case.

THEOREM 3.3. *Consider*

$$\begin{aligned}
\dot{x} &= ax + dy^m + c(x^2) + c(xy) + c(y^{m+1}), \\
\dot{y} &= by^m + c(x^2) + c(xy) + c(y^{m+1}),
\end{aligned} \tag{3.2}$$

where $a \neq 0$, $b \neq 0$, *and* $m \geqq 2$. *If* m *is odd*, $a<0$ *and* $b<0$, *then the zero solution of* (3.2) *is uniformly asymptotically stable*; *otherwise*, *zero is unstable.*

PROOF. Define

$$
\begin{aligned}
&-V(x, y) = ax^2/2 + by^{m+1}/(m+1) + dxy^m \\
\Rightarrow\quad &-\dot{V}(x, y) = ax\dot{x} + by^m\dot{y} + d(\dot{x}y^m + mxy^{m-1}\dot{y}) \\
&\qquad = ax(ax + dy^m + c(x^2) + c(xy) + c(y^{m+1})) \\
&\qquad\quad + by^m(by^m + c(x^2) + c(xy) + c(y^{m+1})) \\
&\qquad\quad + d(axy^m + dy^{2m} + c(x^2y^m) + c(xy^{m+1}) + c(y^{2m+1}) \\
&\qquad\qquad\qquad + mbxy^{2m-1} + c(x^3y^{m-1}) + c(xy^{2m})) \\
&\qquad = a^2x^2 + b^2y^{2m} + d^2y^{2m} + 2adxy^m + c(x^3) \\
&\qquad\quad + c(x^2y) + c(xy^{m+1}) + c(y^{2m+1}).
\end{aligned}
$$

Hence, for small $|x|$ and $|y|$,

$$
\begin{aligned}
-\dot{V}(x, y) &\geqq \rho^2a^2x^2 + \theta b^2y^{2m} + d^2y^{2m} - |2\sigma adxy^m| \\
&= (|\rho ax| - |\sigma dy^m/\rho|)^2 + [\theta b^2 + (1 - \sigma^2/\rho^2)d^2]y^{2m},
\end{aligned}
$$

where ρ, $\theta = 1^-$ and $\sigma = 1^+$. Thus, the coefficient of y^{2m} is positive near the origin and $-\dot{V}(x, y)$ is positive definite. Now suppose m is odd, $a<0$ and $b<0$. Then

$$
\begin{aligned}
V(x, y) &= |a|\, x^2/2 + |b|\, y^{m+1}/(m+1) - dxy^m \\
&= \left(\left(\frac{|a|}{2}\right)^{1/2} x - \frac{dy^m}{(2\,|a|)^{1/2}}\right)^2 + \left(\frac{|b|}{m+1} - \frac{d^2y^{m-1}}{2\,|a|}\right)y^{m+1}
\end{aligned}
$$

so that $V(x, y)$ is positive definite; otherwise, $V(x, y)$ is negative arbitrarily near the origin along at least one of the coordinate axes. □

Unfortunately, the author has not yet been able to answer the question of whether or not the preceding theorem has a natural extension in higher dimensions.

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