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ON THE DEPTH OF TOPOLOGICAL SPACES*

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ABSTRACT. In this short note we prove that Kowalsky's star-space of weight ω_1 is a suitable counterexample to a conjecture of I. Juhàsz.

By a strictly decreasing chain in a topological space X we mean a well-ordered sequence $\{G_\xi;\ \xi<\alpha\}$ of nonempty open subsets of X such that for $\eta<\xi<\alpha$, $\bar{G}_\xi \subsetneqq G_\eta$. The depth of the space X is defined as the supremum of the cardinalities of its strictly decreasing chains and it will be denoted by $k(X)$.

As I. Juhàsz proved [1], for a connected topological space X, $k(X) \leqq (\chi(X))^+$ holds, where $\chi(X)$ denotes the topological character of the space X. He notes also that, as the long line shows, this result is the best possible in the class of connected spaces. However, the long line is not paracompact, hence I. Juhàsz raised the following problem: Is it true that, for a connected paracompact space X, $k(X) \leqq \chi(X)$ holds?

Our aim is to prove that Kowalsky's star-space [2] of weight ω_1 is a counterexample to this conjecture. So our space X is the set $\{0\} \cup \bigcup \{\{\xi\} \times (0, 1];\ \xi<\omega_1\}$ with the topology generated by the metric

$$\rho(0, (\xi, x)) = x, \qquad \rho((\xi, x), (\eta, y)) = |y - x|, \quad \text{if } \xi = \eta,$$
$$= x + y, \quad \text{if } \xi \neq \eta.$$

Now the space X is connected and metrizable hence paracompact and $\chi(X)=\omega_0$. We have only to prove that $k(X)=\omega_1$.

As is well known (see e.g. [3]), the open interval $(0, 1)$ for each $\xi<\omega_1$ contains a sequence $\{x_\eta^\xi;\ \eta<\xi\}$ similar to ξ. Put now

$$G_\xi = \bigcup \{\{\alpha\} \times (x_\xi^\alpha, 1];\ \xi < \alpha < \omega_1\} \qquad (\xi < \omega_1).$$

It is evident that the G_ξ's form a strictly decreasing chain of cardinality ω_1 in X. Q.E.D.

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* This article was not proofread by the author because the Amer. Math. Soc. was unable to locate him. The address given at the end of the paper is the last address given by the author.

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