

ON THE DEPTH OF TOPOLOGICAL SPACES\*

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ABSTRACT. In this short note we prove that Kowalsky's star-space of weight  $\omega_1$  is a suitable counterexample to a conjecture of I. Juhász.

By a strictly decreasing chain in a topological space  $X$  we mean a well-ordered sequence  $\{G_\xi; \xi < \alpha\}$  of nonempty open subsets of  $X$  such that for  $\eta < \xi < \alpha$ ,  $G_\xi \subseteq G_\eta$ . The depth of the space  $X$  is defined as the supremum of the cardinalities of its strictly decreasing chains and it will be denoted by  $k(X)$ .

As I. Juhász proved [1], for a connected topological space  $X$ ,  $k(X) \leq (\chi(X))^+$  holds, where  $\chi(X)$  denotes the topological character of the space  $X$ . He notes also that, as the long line shows, this result is the best possible in the class of connected spaces. However, the long line is not paracompact, hence I. Juhász raised the following problem: Is it true that, for a connected paracompact space  $X$ ,  $k(X) \leq \chi(X)$  holds?

Our aim is to prove that Kowalsky's star-space [2] of weight  $\omega_1$  is a counterexample to this conjecture. So our space  $X$  is the set  $\{0\} \cup \bigcup \{\{\xi\} \times (0, 1]; \xi < \omega_1\}$  with the topology generated by the metric

$$\rho(0, (\xi, x)) = x, \quad \rho((\xi, x), (\eta, y)) = |y - x|, \quad \text{if } \xi = \eta, \\ = x + y, \quad \text{if } \xi \neq \eta.$$

Now the space  $X$  is connected and metrizable hence paracompact and  $\chi(X) = \omega_0$ . We have only to prove that  $k(X) = \omega_1$ .

As is well known (see e.g. [3]), the open interval  $(0, 1)$  for each  $\xi < \omega_1$  contains a sequence  $\{x_\eta^\xi; \eta < \xi\}$  similar to  $\xi$ . Put now

$$G_\xi = \bigcup \{\{\alpha\} \times (x_\eta^\alpha, 1]; \eta < \alpha < \omega_1\} \quad (\xi < \omega_1).$$

It is evident that the  $G_\xi$ 's form a strictly decreasing chain of cardinality  $\omega_1$  in  $X$ . Q.E.D.

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