

## A NOTE ON WALLMAN EXTENDIBLE FUNCTIONS

DARRELL W. HAJEK

**ABSTRACT.** It is known that any continuous function into a  $T_4$  space has a unique continuous Wallman extension, and that any continuous Wallman extension of a continuous function with a  $T_3$  range must be unique. We show that for any  $T_3$  space  $Y$  which is not  $T_4$  there exists a  $T_3$  space  $X$  and a continuous function  $f: X \rightarrow Y$  which has no continuous Wallman extension.

In this paper we will consider only  $T_1$  spaces. In [2] it is shown that if  $Y$  is a  $T_3$  space and  $f: X \rightarrow Y$  is a continuous function having a continuous Wallman extension  $f^*: W(X) \rightarrow W(Y)$  then the extension is unique. Furthermore it follows immediately from the fact that if  $Y$  is  $T_4$  then  $W(Y)$  is  $T_2$  and from the Taimanov theorem (see [1, p. 110]) that if  $Y$  is  $T_4$  then any continuous function  $f: X \rightarrow Y$  has a continuous Wallman extension, and so it is natural to ask whether the condition that  $Y$  be  $T_4$  can be relaxed. In this paper we show that, if consideration is restricted to  $T_3$  spaces, the answer is no.

Recall that for a given space  $X$  the Wallman compactification  $W(X)$  is the collection of all ultrafilters in the lattice of closed subsets of  $X$  given the topology generated by the collection of all sets of the form  $C(A) = \{u \in W(X) : A \in u\}$ , where  $A$  is closed in  $X$  as a base for the closed sets.

The function  $\varphi_X: X \rightarrow W(X)$  defined by  $\varphi_X(x) = \{A : A \text{ closed in } X \text{ and } x \in A\}$  is a dense embedding of  $X$  in  $W(X)$ . A Wallman extension of a function  $f: X \rightarrow Y$  is a function  $f^*: W(X) \rightarrow W(Y)$  such that  $f^* \circ \varphi_X = \varphi_Y \circ f$ .

**THEOREM.** *Let  $Y$  be a  $T_3$  space. Then, unless  $Y$  is  $T_4$ , there is a  $T_3$  space  $X$  and a continuous function  $f: X \rightarrow Y$  which has no continuous Wallman extension.*

**PROOF.** In [2] it was proved that if  $T$  is  $T_3$ , then, given any continuous function  $g: Z \rightarrow T$  which has a continuous Wallman extension  $g^*: W(Z) \rightarrow W(T)$ , for each  $u \in W(Z)$ ,  $\{g^*(u)\} = \bigcap \{C(\text{cl}_T(g[A])) : A \in u\}$ . Suppose now

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that  $Y$  is not  $T_4$ .  $W(Y)$  is not Hausdorff; so there exist two points  $u, v \in W(Y)$  which cannot be separated by disjoint open sets. Let  $\mathcal{O}$  denote the set  $\{\varphi_Y^{-1}[U \cap V]: U, V \text{ open in } W(Y), u \in U, \text{ and } v \in V\}$ . We denote by  $X$  the product space  $\times \{O: O \in \mathcal{O}\}$  and by  $q$  the projection of  $X$  onto  $\varphi_Y^{-1}[W(Y) \cap W(Y)] = Y$ . For each  $P \in \mathcal{O}$  we define  $A(P)$  to be  $\{(y_O)_{O \in \mathcal{O}} \in X: P \subseteq O \Rightarrow y_P = y_O\}$ . It is immediate that  $\{A(P): P \in \mathcal{O}\}$  is a filterbase in the lattice of closed subsets of  $X$ , and, hence, must be contained in some  $w \in W(X)$ . Suppose  $q$  were to have a continuous Wallman extension  $q^*$ . If  $q^*(w) \neq u$  there is some  $K \in w$  such that  $u \notin C(\text{cl}_Y(q[K]))$ . However

$$\varphi_Y^{-1}[W(Y) \sim C(\text{cl}_Y(q[K]))] = Y \sim \text{cl}_Y(q[K]) \in \mathcal{O}$$

and

$$A(Y \sim \text{cl}_Y(q[K])) \subseteq q^{-1}[Y \sim \text{cl}_Y(q[K])];$$

so, since  $A(Y \sim \text{cl}_Y(q[K]))$  must have nonempty intersection with  $K$ ,  $q^*(w)$  must be  $u$ , but, since precisely the same argument can be used to show  $q^*(w) = v$ , we must conclude that  $q$  has no continuous Wallman extension.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PUERTO RICO, MAYAGUEZ, PUERTO RICO 00708