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A NOTE ON WALLMAN EXTENDIBLE FUNCTIONS

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ABSTRACT. It is known that any continuous function into a T_4 space has a unique continuous Wallman extension, and that any continuous Wallman extension of a continuous function with a T_3 range must be unique. We show that for any T_3 space Y which is not T_4 there exists a T_3 space X and a continuous function $f: X \to Y$ which has no continuous Wallman extension.

In this paper we will consider only T_1 spaces. In [2] it is shown that if Y is a T_3 space and $f: X \to Y$ is a continuous function having a continuous Wallman extension $f^*: W(X) \to W(Y)$ then the extension is unique. Furthermore it follows immediately from the fact that if Y is T_4 then $W(Y)$ is T_2 and from the Taimanov theorem (see [1, p. 110]) that if Y is T_4 then any continuous function $f: X \to Y$ has a continuous Wallman extension, and so it is natural to ask whether the condition that Y be T_4 can be relaxed. In this paper we show that, if consideration is restricted to T_3 spaces, the answer is no.

Recall that for a given space X the Wallman compactification $W(X)$ is the collection of all ultrafilters in the lattice of closed subsets of X given the topology generated by the collection of all sets of the form $C(A) = \{u \in W(X) : A \in u\}$, where A is closed in X as a base for the closed sets.

The function $\varphi_X: X \to W(X)$ defined by $\varphi_X(x) = \{A : A$ closed in X and $x \in A\}$ is a dense embedding of X in $W(X)$. A Wallman extension of a function $f: X \to Y$ is a function $f^*: W(X) \to W(Y)$ such that $f^* \circ \varphi_X = \varphi_Y \circ f$.

THEOREM. *Let Y be a T_3 space. Then, unless Y is T_4, there is a T_3 space X and a continuous function $f: X \to Y$ which has no continuous Wallman extension.*

PROOF. In [2] it was proved that if T is T_3, then, given any continuous function $g: Z \to T$ which has a continuous Wallman extension $g^*: W(Z) \to W(T)$, for each $u \in W(Z)$, $\{g^*(u)\} = \bigcap \{C(\mathrm{cl}_T(g[A])) : A \in u\}$. Suppose now

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that Y is not T_4. $W(Y)$ is not Hausdorff; so there exist two points $u, v \in W(Y)$ which cannot be separated by disjoint open sets. Let $\mathcal{O}$ denote the set $\{\varphi_Y^{-1}[U \cap V]: U, V$ open in $W(Y)$, $u \in U$, and $v \in V\}$. We denote by X the product space $\times \{O: O \in \mathcal{O}\}$ and by q the projection of X onto $\varphi_Y^{-1}[W(Y) \cap W(Y)] = Y$. For each $P \in \mathcal{O}$ we define $A(P)$ to be $\{(y_O)_{O \in \mathcal{O}} \in X: P \subseteq O \Rightarrow y_P = y_O\}$. It is immediate that $\{A(P): P \in \mathcal{O}\}$ is a filterbase in the lattice of closed subsets of X, and, hence, must be contained in some $w \in W(X)$. Suppose q were to have a continuous Wallman extension q^*. If $q^*(w) \neq u$ there is some $K \in w$ such that $u \notin C(\mathrm{cl}_Y(q[K]))$. However

$$\varphi_Y^{-1}[W(Y) \sim C(\mathrm{cl}_Y(q[K]))] = Y \sim \mathrm{cl}_Y(q[K]) \in \mathcal{O}$$

and

$$A(Y \sim \mathrm{cl}_Y(q[K])) \subseteq q^{-1}[Y \sim \mathrm{cl}_Y(q[K])];$$

so, since $A(Y \sim \mathrm{cl}_Y(q[K]))$ must have nonempty intersection with K, $q^*(w)$ must be u, but, since precisely the same argument can be used to show $q^*(w) = v$, we must conclude that q has no continuous Wallman extension.

References

1. R. Engelking, *Outline of general topology*, PWN, Warsaw, 1965; English transl., North-Holland, Amsterdam; Interscience, New York, 1968. MR **36** #4508; **37** #5836.

2. D. Hajek, *A characterization of T_3 spaces*, Indiana Univ. Math. J. **23** (1973), 23–25.

3. H. Wallman, *Lattices and topological spaces*, Ann. of Math. **89** (1938), 112–126.

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