A NOTE ON WALLMAN EXTENDIBLE FUNCTIONS

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Abstract. It is known that any continuous function into a $T_4$ space has a unique continuous Wallman extension, and that any continuous Wallman extension of a continuous function with a $T_5$ range must be unique. We show that for any $T_3$ space $Y$ which is not $T_4$ there exists a $T_3$ space $X$ and a continuous function $f: X \to Y$ which has no continuous Wallman extension.

In this paper we will consider only $T_1$ spaces. In [2] it is shown that if $Y$ is a $T_3$ space and $f: X \to Y$ is a continuous function having a continuous Wallman extension $f^*: W(X) \to W(Y)$ then the extension is unique. Furthermore it follows immediately from the fact that if $Y$ is $T_4$ then $W(Y)$ is $T_4$ and from the Taimanov theorem (see [1, p. 110]) that if $Y$ is $T_4$ then any continuous function $f: X \to Y$ has a continuous Wallman extension, and so it is natural to ask whether the condition that $Y$ be $T_4$ can be relaxed. In this paper we show that, if consideration is restricted to $T_3$ spaces, the answer is no.

Recall that for a given space $X$ the Wallman compactification $W(X)$ is the collection of all ultrafilters in the lattice of closed subsets of $X$ given the topology generated by the collection of all sets of the form $C(A) = \{u \in W(X): A \in u\}$, where $A$ is closed in $X$ as a base for the closed sets.

The function $\varphi_X: X \to W(X)$ defined by $\varphi_X(x) = \{A: A$ closed in $X$ and $x \in A\}$ is a dense embedding of $X$ in $W(X)$. A Wallman extension of a function $f: X \to Y$ is a function $f^*: W(X) \to W(Y)$ such that $f^* \circ \varphi_X = \varphi_Y \circ f$.

Theorem. Let $Y$ be a $T_3$ space. Then, unless $Y$ is $T_4$, there is a $T_3$ space $X$ and a continuous function $f: X \to Y$ which has no continuous Wallman extension.

Proof. In [2] it was proved that if $T$ is $T_3$, then, given any continuous function $g: Z \to T$ which has a continuous Wallman extension $g^*: W(Z) \to W(T)$, for each $u \in W(Z)$, $\{g^*(u)\} = \bigcap \{C(\text{cl}_T(g[A])): A \in u\}$. Suppose now...
that $Y$ is not $T_4$. $W(Y)$ is not Hausdorff; so there exist two points $u, v \in W(Y)$ which cannot be separated by disjoint open sets. Let $\mathcal{O}$ denote the set $\{\varphi_Y^{-1}[U \cap V] : U, V \text{ open in } W(Y), u \in U, \text{ and } v \in V\}$. We denote by $X$ the product space $\prod \{O : O \in \mathcal{O}\}$ and by $q$ the projection of $X$ onto $\varphi_Y^{-1}[W(Y) \cap W(Y)] = Y$. For each $P \in \mathcal{O}$ we define $A(P)$ to be $\{(y_0)_{O \in \mathcal{O}} \in X : P \subseteq O \Rightarrow y_0 \in \varphi_Y^{-1}(P)\}$. It is immediate that $\{A(P) : P \in \mathcal{O}\}$ is a filterbase in the lattice of closed subsets of $X$, and, hence, must be contained in some $w \in W(X)$. Suppose $q$ were to have a continuous Wallman extension $q^*$. If $q^*(w) \neq u$ there is some $K \in w$ such that $u \notin \text{cl}_Y(q[K])$. However

$$
\varphi_Y^{-1}[W(Y) \sim \text{cl}_Y(q[K])] = Y \sim \text{cl}_Y(q[K]) \in \mathcal{O}
$$

and

$$
A(Y \sim \text{cl}_Y(q[K])) \subseteq \varphi_Y^{-1}[Y \sim \text{cl}_Y(q[K])];
$$

so, since $A(Y \sim \text{cl}_Y(q[K]))$ must have nonempty intersection with $K$, $q^*(w)$ must be $u$, but, since precisely the same argument can be used to show $q^*(w) = v$, we must conclude that $q$ has no continuous Wallman extension.

REFERENCES


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