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A CATEGORY ANALOGUE OF THE HEWITT-SAVAGE ZERO-ONE LAW

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ABSTRACT. A category analogue of the Hewitt-Savage zero-one law is obtained in this paper.

Oxtoby [**3**, (3.1), p. 162] obtained a category analogue of the Kolmogorov zero-one law for measures. The main aim of this article is to obtain a category analogue of the Hewitt-Savage zero-one law. The desired result follows as a simple consequence of the theorem we prove here which is of independent interest. For details concerning the Hewitt-Savage zero-one law for measures see Meyer [**1**, T4 Corollary, p. 150].

A topological space X is said to be a Baire space if no nonempty open subset of X is of first category. A family $\mathscr{B}$ of nonempty open sets in a topological space is called a pseudo-base if every nonempty open set contains an element of $\mathscr{B}$. In any Baire space countable intersections of dense open sets are dense. Let X be any set and G a group (under composition) of transformations on X to X. A subset A of X is said to be invariant if $TA=A$ for every T in G. For any x in X, the orbit of x is defined to be the set $\{Tx; T\in G\}$. A subset D of a topological space is said to have the property of Baire if we can write $D=E\,\Delta\, P$, where E is any open subset of X and P is a set of first category in X. A topological space X is said to be homogeneous if for any two points x and y in X there exists a homeomorphism T from X to X such that $Tx=y$.

THEOREM. *Let X be a topological space and G a group* (*under composition*) *of homeomorphisms on X. In the following* (1)$\Rightarrow$(2)$\Leftrightarrow$(3)$\Rightarrow$(4). *If X is a Baire space, then* (4)$\Rightarrow$(2). *If, further, X has a countable pseudo-base, then* (4)$\Rightarrow$(1).

(1) *The orbit of some point x in X is dense in X.*

(2) *For any two nonempty open sets U and V, $TU\cap V\neq\varnothing$ for some T in G.*

(3) *Any nonempty open invariant subset of X is dense in X.*

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(4) *Any invariant subset D of X with the property of Baire is either of first category or $X-D$ is of first category in X.*

PROOF. (1)$\Rightarrow$(2)$\Leftrightarrow$(3) are easy to prove.

(3)$\Rightarrow$(4). Since (2) and (3) are equivalent, we use (2). Suppose D is an invariant subset of X with the property of Baire. Then $D=(V-P)\cup Q$, where V is open in X, and P, Q are sets of the first category in X. Suppose $V\neq\varnothing$ and let U be any nonempty open subset of X. By (2), there is a T in G such that $TV\cap U\neq\varnothing$. Then $(X-D)\cap(TV\cap U)\subset TP$. Since T is a homeomorphism, TP is of the first category in X. So, we have proved that given any nonempty open set U in X there is a nonempty open set U^* $(=TV\cap U)$ contained in U such that $U^*\cap(X-D)$ is of the first category in X. By the Banach category theorem [**4**, Theorem 16.1, p. 62], $X-D$ is of the first category in X.

(4)$\Rightarrow$(2). Let X be a Baire space. Let U and V be two nonempty open subsets of X. $\bigcup_{T\in G} TU$ is an open invariant subset of X. By (4) and the fact that X is a Baire space it follows that $X-\bigcup_{T\in G} TU$ is of the first category. Since X is a Baire space $\bigcup_{T\in G} TU$ is dense in X. Hence $TU\cap V\neq\varnothing$ for some T in G.

(4)$\Rightarrow$(1). Let X be a Baire space with a countable pseudo-base $A_1, A_2, \cdots$. It is easy to verify that $\{x \text{ in } X\text{:orbit of } x \text{ is dense in } X\}=\bigcap_{n\geq 1}\bigcup_{T\in G} TA_n$. Since X is a Baire space, $\bigcup_{T\in G} TA_n$ is dense open in X for every $n\geq 1$. Consequently, $\bigcap_{n\geq 1}\bigcup_{T\in G} TA_n$ is a dense subset of X.

For a result similar to our theorem see also p. 443 of [**2**].

We obtain the category analogue of the Hewitt-Savage zero-one law as a corollary to the preceding theorem.

COROLLARY 1. *Let X be any topological space. Let J be any infinite set. Let $Y=\prod_{\alpha\in J} X_\alpha$ equipped with the product topology where each $X_\alpha=X$. Let G be the group of all transformations T from Y to Y, where each T permutes finitely many coordinates of the points of Y leaving the rest the same. Then any G-invariant subset D of Y with the property of Baire is either of first category or its complement is of first category. In particular, if D is a tail subset of Y with the property of Baire then either D or $Y-D$ is of first category.*

PROOF. Check that condition (2) of the Theorem is satisfied for basic open subsets of X^J. The second part of the corollary follows from the fact that any tail set is G-invariant. For the definition of a tail set see Oxtoby [**4**, p. 84].

REMARK. Oxtoby [**3**, (3.1), p. 162] obtained the second part of Corollary 1 for any family of topological spaces $\{X_\alpha:\alpha\in J\}$ where each X_α admits a countable pseudo-base. In the special case where each $X_\alpha=X$, Corollary 1 is an improvement of Oxtoby's result. It would be interesting to prove Oxtoby's result dropping the countable pseudo-base condition.

COROLLARY 2 (H. E. WHITE, JR.). *If, for each* α *in* J, X_α *is a homogeneous space and* D *is a tail set with the property of Baire in* $Y=\prod_{\alpha\in J} X_\alpha$, *then either* D *or* $Y-D$ *is of first category in* Y.

PROOF. Let $\mathscr{G}_\alpha$ be the group of all homeomorphisms on X_α, $\alpha\in J$. Let I_α denote the identity transformation on X_α. Let G be the group of all transformations $\prod_{\alpha\in J} G_\alpha$ from Y to Y where each $G_\alpha\in\mathscr{G}_\alpha$ and $G_\alpha=I_\alpha$ for all but a finite number of α in J. It is easy to check that with respect to this group G, condition (2) of the Theorem is satisfied and any tail set in Y is G-invariant.

REMARK. The referee suggested Corollary 2 as an application of the Theorem. He points out that H. E. White, Jr. [**5**] proved this corollary in a different fashion.

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