PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

LOCALIZING EQUIVARIANT BORDISM

E. R. WHEELER

ABSTRACT. The unitary bordism of a finite group is computed up to torsion and an equivariant Rohlin exact sequence is exhibited for groups of odd order.

1. **Introduction.** In [4] Erich Ossa shows that some questions in equivariant bordism can be answered easily when one tensors with the appropriate subring of the rationals. The purpose of this note is to use this technique to understand several facts about equivariant bordism theories. Let $\Omega_*^O(G)$, $\Omega_*^{SO}(G)$, and $\Omega_*^U(G)$ denote respectively the unoriented, oriented, and unitary bordism of a finite group G. Among the results recorded in this paper are:

PROPOSITION 1. *If the order of G is d, then $\Omega_*^U(G)\otimes R_d$ is a free $\Omega_*^U\otimes R_d$ module on even dimensional generators (where $R_d=\{a/b|a$ is any integer and b is a power of $d\}$).*

PROPOSITION 3. *$\Omega_*^U(G)\otimes_{\Omega_*^U}\Omega_*^O\to\Omega_*^O(G)$ is epic if the order of G (denoted $|G|$) is odd.*

PROPOSITION 5. *$\Omega_*^{SO}(G)\otimes_{\Omega_*^{SO}}\Omega_*^O\to\Omega_*^O(G)$ is an isomorphism if $|G|$ is odd.*

PROPOSITION 7. *If $|G|$ is odd, there is an equivariant Dold exact triangle*

$$\begin{tikzcd} \Omega_*^{SO}(G)\oplus\Omega_*^O(G) \arrow[r, "{(2,0)}"] & \Omega_*^{SO}(G) \arrow[r, "\rho"] & \Omega_*^O(G). \arrow[ll, bend left] \end{tikzcd}$$

2. **Preliminaries.** It is necessary to consider bordism of 4-tuples $[M, \partial M, \phi, f]$ where

(i) M and ∂M are compact differentiable manifolds where ∂M is the boundary of M.

(ii) $\phi: G\times M\to M$ is a differentiable G action.

(iii) The isotropy subgroups of points in M and ∂M are respectively elements of the families $\mathscr{F}$ and $\mathscr{F}'$ of subgroups of G.

(iv) $f: M\to X$ is an equivariant map into the G space X.

Received by the editors May 22, 1973 and, in revised form, June 19, 1973.
AMS (MOS) subject classifications (1970). Primary 57D85.
Key words and phrases. Equivariant bordism group, Rohlin exact sequence.

As in [7, §2], the bordism equivalence classes of such objects, $\Omega_*^O(G)(\mathscr{F}, \mathscr{F}')(X)$, form an Ω_*^O module. If the manifold M has a stable complex structure or an orientation on its tangent bundle which is preserved by the group action, the corresponding bordism theories are denoted respectively by $\Omega_*^U(G)(\mathscr{F}, \mathscr{F}')(X)$ and $\Omega_*^{SO}(G)(\mathscr{F}, \mathscr{F}')(X)$ (see Stong [5]). For $L=O$, SO, and U and families of subgroups $\mathscr{F}' \subset \mathscr{F}$, there is an exact triangle

$$\begin{tikzcd} \Omega_*^L(G)(\mathscr{F}')(X) \arrow[r, "i_*"] & \Omega_*^L(G)(\mathscr{F})(X) \arrow[r, "j_*"] & \Omega_*^L(G)(\mathscr{F}, \mathscr{F}')(X) \arrow[ll, bend left, "\partial"] \end{tikzcd}$$

where i_* and j_* forget respectively $\mathscr{F}'$ and $\mathscr{F}$ freeness and where ∂ restricts to the boundary.

One notes that, as in [5, §3], there are universal classifying spaces for real or complex n-plane bundles with G action, denoted here by BO_n or BU_n. For $L=O$ or U denote by $F'_K(BL_n)$ the components of the fixed set of K acting on BL_n above which the fibers of the canonical bundle have no trivial irreducible summands when decomposed as K-spaces. From Stong's Proposition 5.1 [7] it follows that if $\mathscr{F}' \subset \mathscr{F}$ are adjacent families (i.e. $\mathscr{F} - \mathscr{F}' = \{K\}$, the elements of the conjugacy class of a subgroup K), then

$$\Phi_K : \Omega_*^L(G)(\mathscr{F}, \mathscr{F}')(X) \to \bigoplus_s \Omega_*^L(N_G(K)/K)(\{\{1\}\})(F_K(X) \times F'_K(BL_s))$$

is an isomorphism where Φ_K is defined via the fixed set information of K. The fixed set of a subgroup $K<G$ acting on BU_n is a union of products of BU_t's. If K is a group of odd order, each nontrivial irreducible real representation is the realification of an irreducible complex representation. Hence in this case, $F'_K(BO_n)$ is a union of products of BU_t's. For an n-tuple $(t_1, \cdots, t_n) = (t)$, let $BU_{(t)}$ denote the product of G-spaces BU_{t_i}. Then

$$\Omega_*^L(G)(\mathscr{F}, \mathscr{F}')(BU_{(t)}) \simeq \bigoplus \Omega_*^L(N_G(K)/G)(\{\{1\}\})(BU_{(s)})$$

for all G if $L=U$, and for G of odd order if $L=O$ or SO.

Suppose $|G|=d$. The well-known transfer map tells one that, if $\pi : X \to X/G$ is the quotient map which identifies the G orbits and if $H_*(X; R_d)$ is a free R_d module, then $\pi_* : \mathrm{INV}(H_*(X; R_d)) \to H_*(X/G; R_d)$ is an isomorphism where INV( ) denotes the submodule of elements invariant under the G action [3, Corollary 2.3]. It follows then from the collapse of the bordism spectral sequence for $\Omega_*^U(\ ) \otimes R_d$ in this instance that $\Omega_*^U(X/G) \otimes R_d \simeq \Omega_*^U \otimes \mathrm{INV}(H_*(X; R_d))$.

3. **Unitary equivariant bordism.** Note that if $\mathscr{F}$ is the family of all subgroups of G, then $\Omega_*^U(G)(\mathscr{F})(\mathrm{pt.}) = \Omega_*^U(G)$ discussed in the introduction.

One then has

PROPOSITION 1′. *For any family of subgroups of* G, $\Omega_*^U(G)(\mathscr{F})(BU_{(t)})\otimes R_d$ *is a free* $\Omega_*^U\otimes R_d$ *module on even dimensional generators for any finite group* G.

PROOF. One inducts on the order of G and the size of the family $\mathscr{F}$. If $\mathscr{F}=\{\{1\}\}$, then [7, Proposition 5.2] yields that $\Omega_*^U(G)(\{\{1\}\})(BU_{(t)})\otimes R_d\cong\Omega_*^U(BU_{(t)}\times_G EG)\otimes R_d\cong\Omega_*^U\otimes\mathrm{INV}(H_*(BU_{(t)};R_d))$ where EG is the total space of a universal principal G bundle and G acts diagonally on $BU_{(t)}\times EG$.

To complete the induction step, choose a maximal subgroup K of a family $\mathscr{F}$. Let $\mathscr{F}'=\mathscr{F}-\{K\}$. One has the long exact sequence

$$\begin{tikzcd}
\Omega_*^U(G)(\mathscr{F}')(BU_{(t)})\otimes R_d \arrow[r] & \Omega_*^U(G)(\mathscr{F})(BU_{(t)})\otimes R_d \arrow[r] & \bigoplus \Omega_*^U(N_G(K)/K)(\{\{1\}\})(BU_{(t)})\otimes R_d \arrow[ll, bend left]
\end{tikzcd}$$

where the first and third modules are free $\Omega_*^U\otimes R_d$ modules on even dimensional generators. Since $\Omega_{\mathrm{odd}}^U=0$, the long sequence is in fact a split short exact sequence and the result follows. □

There is the fixed point set map

$$\Phi_{\mathscr{F}}:\Omega_*^U(G)(\mathscr{F})\to\bigoplus_{\{H\}\subset\mathscr{F}}\ \bigoplus_{(t)}\Omega_*^U(BU_{(t)})$$

defined by picking a representative of each conjugacy class $\{H\}\subset\mathscr{F}$, considering the fixed set of each representative H, decomposing the normal bundle to the fixed set via the nontrivial representations of H, and classifying the pieces into the appropriate BU_t. One sums over the representatives and the components of the fixed sets to complete the definition of $\Phi_{\mathscr{F}}$ (see [7, §12]).

PROPOSITION 2. $\Phi_{\mathscr{F}}:\Omega_*^U(G)(\mathscr{F})\otimes R_d\to\bigoplus\Omega_*^U(BU_{(t)})\otimes R_d$ *is monic for all families.*

PROOF. If $\mathscr{F}=\{\{1\}\}$, then $\Omega_*^U(G)(\{\{1\}\})=\tilde{\Omega}_*^U(BG)\oplus\Omega_*^U$ and the reduced bordism of BG consists of d torsion. Letting $\mathscr{F}'\subset\mathscr{F}$ be adjacent families differing by $\{K\}$, one has:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_*^U(G)(\mathscr{F}')\otimes R_d \arrow[r] \arrow[dd, "\Phi_{\mathscr{F}'}"] & \Omega_*^U(G)(\mathscr{F})\otimes R_d \arrow[r, "\Phi_K\circ j_*"] \arrow[dd, "\Phi_{\mathscr{F}}"] & \bigoplus\Omega_*^U(N_G(K)/K)(\{\{1\}\})(BU_{(t)})\otimes R_d \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\
 & & & \bigoplus(\Omega_*^U\otimes\mathrm{INV}(H_*(BU_{(t)};R_d))) \arrow[d] & \\
0 \arrow[r] & \bigoplus\limits_{\{H\}\subset\mathscr{F}'}\bigoplus\Omega_*^U(BU_{(t)})\otimes R_d \arrow[r] & \bigoplus\limits_{\{H\}\subset\mathscr{F}}\bigoplus\Omega_*^U(BU_{(t)})\oplus R_d \arrow[r] & \bigoplus\Omega_*^U(BU_{(t)})\otimes R_d \arrow[r] & 0
\end{tikzcd}$$

The diagrams commute, the short exact sequences are split exact, the third map is monic and one may assume the first map is monic by induction. Hence the second map is monic. □

One notes that for a group of odd order the nontrivial irreducible complex representations have the property that a representation is not equivalent to its conjugate [9, Theorem 4.7.3]. Let $I(K)$ be the nontrivial irreducible complex representations of K. $N_G(K)/K$ acts on $I(K)$ with $(g,\pi)=\pi^g$ where $\pi^g(k)=\pi(g^{-1}kg)$ for $k\in K$. It follows that $\pi^g\neq\bar\pi$ for any $g\in N_G(K)/K$ so that $I(K)=A\cup A'$ where A and A' are disjoint $N_G(K)/K$ invariant sets with $A'=\{\bar\pi\mid\pi\in A\}$. Hence the $N_G(K)/K$ space $F'_K(BU_t)$ decomposes into $N_G(K)/K$ invariant pieces $F'_{K,A}(BU_t)\cup D$ where D consists of those components of $F'_K(BU_t)$ such that the K decomposition of the fibers of the canonical bundle contains at least one summand from A'. Now it is clear that the realification map r: $F'_{K,A}(BU_t)\to F'_K(BO_{2t})$ is a homotopy equivalence and is equivariant. Hence the map induced by $r\times 1$ from

$$F'_{K,A}(BU_t)\underset{N_G(K)/K}{\times}E(N_G(K)/K)\to F'_K(BO_{2t})\underset{N_G(K)/K}{\times}E(N_G(K)/K)$$

is a homotopy equivalence. Then one has:

PROPOSITION 3′. *If $|G|=d$ is odd, $\rho:\Omega^U_*(G)(\mathscr{F})\otimes_{\Omega_*U}\Omega^O_*\to\Omega^O_*(G)(\mathscr{F})$ is epic for all families $\mathscr{F}$ where ρ is induced by the map which forgets complex structure.*

PROOF. One inducts as before. For $\mathscr{F}'\subset\mathscr{F}$ adjacent families one has:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^U_*(G)(\mathscr{F}')\otimes R_d \arrow[r]\arrow[d, "\rho"] & \Omega^U_*(G)(\mathscr{F})\otimes R_d \arrow[r]\arrow[d, "\rho"] & \bigoplus \Omega^U_*\big(F'_K(BU_t)\underset{N_G(K)/K}{\times}E(N_G(K)/K)\big) \arrow[r]\arrow[d, "\rho"] & 0\\
& \Omega^O_*(G)(\mathscr{F}') \arrow[r] & \Omega^O_*(G)(\mathscr{F}) \arrow[r] & \bigoplus \Omega^O_*\big(F'_K(BO_{2t})\underset{N_G(K)/K}{\times}E(N_G(K)/K)\big) \arrow[ll, bend left]
\end{tikzcd}$$

The top sequence is split exact so tensoring with $\Omega^O_*\otimes R_d$ over $\Omega^U_*\otimes R_d$ preserves exactness and yields

$$\begin{tikzcd}
0 \arrow[r] & \Omega^U_*(G)(\mathscr{F}')\underset{\Omega_*{}^U}{\otimes}\Omega^O_* \arrow[r]\arrow[d, "\rho"] & \Omega^U_*(G)(\mathscr{F})\underset{\Omega_*{}^U}{\otimes}\Omega^O_* \arrow[r]\arrow[d, "\rho"] & \bigoplus \Omega^O_*\big(F'_K(BU_t)\underset{N_G(K)/K}{\times}E(N_G(K)/K)\big) \arrow[r]\arrow[d, "r\times 1"] & 0\\
0 \arrow[r] & \Omega^O_*(G)(\mathscr{F}') \arrow[r] & \Omega^O_*(G)(\mathscr{F}) \arrow[r] & \bigoplus \Omega^O_*\big(F'_K(BO_{2t})\underset{N_G(K)/K}{\times}E(N_G(K)/K)\big) \arrow[r] & 0
\end{tikzcd}$$

The first and third maps are epic by induction and the comments preceding this proposition, the top sequence is split exact, and this completes the proof. □

NOTE. Using the same arguments one can show that

$$\rho:(\Omega_*^U(G)(\mathscr{F})\otimes R_d)\underset{\Omega_*^U\otimes R_d}{\otimes}(\Omega_*^{SO}\otimes R_d)\to\Omega_*^{SO}(G)(\mathscr{F})\otimes R_d$$

is epic.

4. Oriented equivariant bordism.

PROPOSITION 4. *If $|G|=d$ is odd, then $\Omega_*^{SO}(G)(\mathscr{F})\otimes R_d$ is a free $\Omega_*^{SO}\otimes R_d$ module on even dimensional generators.*

PROOF. One follows the arguments of Proposition 1′ while noting that the comment following Proposition 3′ shows the exact sequence to be split. □

NOTE. If G is abelian, Propositions 1 and 4 are somewhat weaker than previously known results (see [4] and [8]).

PROPOSITION 5′. *If $|G|=d$ is odd then*

$$\Omega_*^{SO}(G)(\mathscr{F})\underset{\Omega_*^{SO}}{\otimes}\Omega_*^O\to\Omega_*^O(G)(\mathscr{F})$$

is an isomorphism for all families $\mathscr{F}$.

PROOF. The proof follows as in Proposition 3′.

There is the standard Rohlin exact sequence $\Omega_*^{SO}\xrightarrow{2}\Omega_*^{SO}\xrightarrow{\rho}\Omega_*^O$ where ρ forgets the orientation. It was observed by Dold that the Rohlin sequence fits into an exact triangle

$$\begin{tikzcd}
\Omega_*^{SO} \arrow[r, "\rho"] & \Omega_*^O \arrow[r] & \Omega_*^{SO}\oplus\Omega_*^O \arrow[ll, bend left, "(2,0)"]
\end{tikzcd}$$

(see [6, p. 216]). Using Proposition 5 one is able to arrive at equivariant versions of these exact sequences.

PROPOSITION 6. *If $|G|=d$ is odd, there is an equivariant Rohlin exact sequence,*

$$\Omega_*^{SO}(G)\xrightarrow{2}\Omega_*^{SO}(G)\xrightarrow{\rho}\Omega_*^O(G).$$

PROOF. Defining $\sigma:B\to B\otimes R_d$ by $\sigma(b)=b\otimes 1$, one has

$$\begin{tikzcd}
\Omega_*^{SO}(G) \arrow[r, "2"] \arrow[d, "\sigma"] & \Omega_*^{SO}(G) \arrow[d, "\sigma"] \arrow[dr, "\rho"] & \\
 & & \Omega_*^O(G)\cong\Omega_*^O(G)\otimes R_d \\
\Omega_*^{SO}(G)\otimes R_d \arrow[r, "2"] & \Omega_*^{SO}(G)\otimes R_d \arrow[ur, "\rho'"] &
\end{tikzcd}$$

in which the bottom sequence is exact since it results from tensoring the standard Rohlin exact sequence by the free module $\Omega_*^{SO}(G)\otimes R_d$. If $\rho(x)=0$ then $\rho'\sigma(x)=0$ so there is $y'\in\Omega_*^{SO}(G)\otimes R_d$ with $2y'=\sigma(x)$. There is $y\in\Omega_*^{SO}(G)$ with $\sigma(y)=d^ty'$ so $\sigma(2y-d^tx)=0$. The kernel of σ is d-torsion so $2d^sy=d^{t+s}x$ or $d^{t+s}x$ is in the image of 2. Since d is odd, x is in the image of 2. Clearly $\rho\circ 2=0$. $\square$

NOTE. P.E. Conner has shown $\Omega_*^{SO}(Z_2)\xrightarrow{2}\Omega_*^{SO}(Z_2)\xrightarrow{\rho}\Omega_*^{O}(Z_2)$ is not exact [1, Theorem 5.8].

PROPOSITION 7. *If $|G|=d$ is odd, there is an equivariant Dold exact triangle*

$$\begin{tikzcd} \Omega_*^{SO}(G)\oplus\Omega_*^{O}(G) \arrow[r, "(2{,}0)"] & \Omega_*^{SO}(G) \arrow[r, "\rho"] & \Omega_*^{O}(G) \arrow[ll, bend left, "\alpha"] \end{tikzcd}$$

induced by tensoring the standard Dold exact triangle with $\Omega_*^{SO}(G)$ over Ω_*^{SO}.

PROOF. One tensors the above triangle with R_d (thus insuring exactness) and compares the two via the homomorphism $\sigma: B\to B\otimes R_d$ as in Proposition 6. Exactness at $\Omega_*^{SO}(G)$ is the Rohlin exact sequence. Exactness at $\Omega_*^{O}(G)$ follows since $\Omega_*^{O}(G)$ is a Z_2 vector space. $(2,0)\circ\alpha=0$ since the image of α is torsion of order 2. Kernel of $(2, 0)$ is the torsion of order 2 and hence by comparison via σ is in the image of α.

NOTE. In the Dold exact triangle the map $\Omega_*^{O}\to\Omega_*^{SO}\oplus\Omega_*^{O}$ is defined by taking the dual submanifolds of the first and second powers of the first Stiefel-Whitney class. In the equivariant situation due to the lack of equivariant transverse regularity the map appears to be purely algebraic.

REFERENCES

1. P. E. Conner, "Orientation preserving involutions", in *Lectures on the action of a finite group*, Lecture Notes in Math., no. 73, Springer-Verlag, Berlin and New York, 1968, Chap. 2. MR **41** #2670.

2. P. E. Connor and E. E. Floyd, *Differentiable periodic maps*, Ergebnisse der Math. und ihrer Grenzgebiete, Band 33, Springer-Verlag, Berlin; Academic Press, New York, 1964. MR **31** #750.

3. E. E. Floyd, *Periodic maps via Smith theory*, Ann. of Math. Studies, no. 46, Chapter III, Princeton Univ. Press, Princeton, N.J., 1960.

4. E. Ossa, *Unitary bordism of abelian groups*, Proc. Amer. Math. Soc. **33** (1972), 568–571. MR **45** #2743.

5. R. E. Stong, *Complex and oriented equivariant bordism*, Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969), Markham, Chicago, Ill., 1970, pp. 291–316. MR **42** #8521.

6. ———, *Notes on cobordism theory*, Mathematical Notes, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1968. MR **40** #2108.

7. ———, *Unoriented bordism and actions of finite groups*, Mem. Amer. Math. Soc., no. 103 (1970). MR **42** #8522.

8. E. R. Wheeler, *The oriented bordism of cyclic groups*, Dissertation, University of Virginia, Charlottesville, Va., 1973.

9. J. A. Wolf, *Spaces of constant curvature*, Chapter 4, McGraw-Hill, New York, 1967. MR **36** #829.

Department of Mathematics, University of Virginia, Charlottesville, Virginia 22901

Current address: Department of Mathematics, Northern Kentucky State College, Highland Heights, Kentucky 41076