PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

ALMOST σ-DEDEKIND COMPLETE RIESZ SPACES AND THE MAIN INCLUSION THEOREM

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ABSTRACT. A Riesz space is *almost σ-Dedekind complete* if it can be embedded as a super order dense Riesz subspace of a σ-Dedekind complete space. The location of this concept in the Main Inclusion Theorem of Luxemburg and Zaanen is investigated; it is shown that this concept is implied by σ-Dedekind completeness and by the projection property, that it implies the property of being Archimedean, and that it is independent of the principal projection property and the property of having sufficiently many projections.

1. **Introduction.** For notation and basic terminology concerning Riesz spaces, we refer the reader to [**4**]. Suppose that L_ρ is a seminormed Riesz space with the σ-Fatou property, and that A is an ideal of L. While investigating the relationship of the properties of the seminorm on L to those of the quotient seminorm on L/A, the authors have found that σ-Dedekind completeness of L is quite useful, but that a weaker property, called almost σ-Dedekind completeness, will often serve equally as well. (The idea is based on the fact that if L_ρ is almost σ-Dedekind complete and has the σ-Fatou property, then it can be embedded nicely in a σ-Dedekind complete space L^* with a seminorm ρ^* which extends ρ, and which also has the σ-Fatou property; one can prove things in L^* and then drop back down to L. See [**1**] and [**2**].) Since this property fits neatly into the Main Inclusion Scheme of Luxemburg and Zaanen [**4**, pp. 137, 174] and since it seems to be of some independent interest, this paper was written.

2. **Almost σ-Dedekind completeness.** We recall that a Riesz subspace L of a Riesz space L^* is *super order dense* in L^* if for every $\theta<u^*\in L^*$ there exists a sequence $\{u_n\}$ of elements of L such that $\theta\leq u_n\uparrow u^*$ in L^*. That is, every positive element of L^* is a *σ-upper element* of L.

DEFINITION. A Riesz space L is *almost σ-Dedekind complete* if it can be embedded as a super order dense Riesz subspace of a σ-Dedekind complete Riesz space L^*. (By "embedding," we mean that the embedding function $\phi: L\to L^*$ is a one-to-one Riesz homomorphism; since the image $\phi(L)$ in L^* is order dense, this implies that ϕ is a normal integral.)

Presented to the Society, January 16, 1974; received by the editors August 6, 1973.
AMS (MOS) subject classification (1970). Primary 46A40.
Key words and phrases. Almost σ-Dedekind complete Riesz spaces, Main Inclusion Theorem, σ-Dedekind completion.

It is obvious that every almost σ-Dedekind complete Riesz space is necessarily Archimedean and that every σ-Dedekind complete Riesz space is necessarily almost σ-Dedekind complete. Further, every order separable Archimedean space is almost σ-Dedekind complete since it is super order dense in its Dedekind completion. Also, if L is almost σ-Dedekind complete, we can always assume without loss of generality that the ideal generated in L^* by L is L^* itself; this implies that every positive element of L^* is also a σ-lower element of L, for if $\theta \leqq u^* \in L^*$, then there exists an element $u \in L$ such that $u^* \leqq u$. It follows that there exists a sequence $\{u_n\}$ of elements of L such that $\theta \leqq u_n \uparrow u - u^*$ in L^* so that if $v_n = u - u_n \in L$, then $v_n \downarrow u^*$ in L^*, and u^* is a σ-lower element of L.

For a fixed Riesz space L^* and a subset K of L^*, we denote by $l(K)$ and $u(K)$ the set of all σ-lower and σ-upper elements of K in L^*. The above discussion shows that if L is almost σ-Dedekind complete, then there exists a σ-Dedekind complete Riesz space L^* such that L is embedded in L^* and $(L^*)^+ = u(L^+) = l(L^+)$. Quinn [5, Corollary 2.4], has shown that if L is an arbitrary Archimedean Riesz space, then there exists an essentially unique "smallest" σ-Dedekind complete Riesz space L^σ, called the *σ-Dedekind completion* of L, in which L can be embedded as an order dense Riesz subspace and which has the property that the ideal generated by L in L^σ is L^σ itself. If L is almost σ-Dedekind complete, we can always take $L^* = L^\sigma$; almost σ-Dedekind complete spaces are thus distinguished from arbitrary Archimedean Riesz spaces by the fact that $(L^\sigma)^+ = u(L^+) = l(L^+)$. (We remark that in general L^σ may be different from the Dedekind completion L^δ.)

Quinn has also introduced independently of us the concept of what he calls an *almost σ-complete* Riesz space which agrees with our definition of almost σ-Dedekind completeness in the case that L is Archimedean. He provides [5, Theorem 8.3 and Corollary 8.5] a characterization of almost σ-complete spaces which for our purposes can be stated as follows:

THEOREM 1. *The Riesz space L is almost σ-Dedekind complete if and only if every order bounded monotone sequence is order Cauchy.*

PROOF. Because Quinn's work is done in more generality than we need, and because the proof is much more straightforward in the case that L is Archimedean, we sketch a proof here. (The argument is basically Quinn's.) We recall that the sequence $\{f_n\}$ is *order Cauchy* if there exists a sequence $u_n \downarrow \theta$ such that $|f_n - f_{n+k}| \leqq u_n$ for $n, k = 1, 2, \cdots$.

Suppose first that L is almost σ-Dedekind complete. It suffices to show that every sequence $u_n \downarrow \geqq \theta$ is order Cauchy. Embed L in a σ-Dedekind complete Riesz space L^* with $(L^*)^+ = u(L^+) = l(L^+)$, and suppose that

$\{u_n\}$ is a sequence of elements of L with $u_n \downarrow \geqq \theta$. Then $u_n \downarrow u^* \geqq \theta$ for some $u^* \in L^*$ and by assumption $\theta \leqq v_n \uparrow u^*$ with $v_n \in L$. If $w_n = u_n - v_n$ for $n = 1, 2, \cdots$ then it is easy to see that $w_n \downarrow \theta$ and that $\theta \leqq u_n - u_{n+k} \leqq w_n$ for all n, k.

Conversely, suppose that every order bounded monotone sequence in L is order Cauchy. We note first that L must be Archimedean, so we can embed L in its Dedekind completion L^δ. We shall show that $l(L^+) \subseteq u(L^+)$; a similar argument shows that $u(L^+) \subseteq l(L^+)$. If we let $L^* = u(L^+) - u(L^+)$, it follows easily that L^* is σ-Dedekind complete and that $(L^*)^+ = u(L^+) = l(L^+)$ implying that L is almost σ-Dedekind complete.

To this end, suppose that $\theta \leqq u^* \in l(L^+)$ so that $u_n \downarrow u^*$ for some sequence $\{u_n\}$ of positive elements of L. By assumption, $\{u_n\}$ is order Cauchy in L, so that $\theta \leqq u_n - u_{n+k} \leqq v_n$ for some sequence $\{v_n\}$ of elements of L with $v_n \downarrow \theta$. Let $w_n = \bigvee_{k=1}^{n} (u_k - v_k)^+ \in L$; then $\theta \leqq w_n \uparrow \leqq u^*$. Since $\{u_n - v_n\}$ order converges to u^* in L^δ, it follows that $\theta \leqq w_n \uparrow u^*$ and thus $u^* \in u(L^+)$. This completes the proof of the theorem. (Note that L^* is actually the σ-Dedekind completion of L.)

Quinn also provides the following sufficient (but not necessary) condition that a Riesz space be almost σ-Dedekind complete: If L has the principal projection property (and is hence Archimedean) and if the lattice of principal bands of L is σ-Dedekind complete, then L is almost σ-Dedekind complete.

We come now to the principal result of the paper, which shows the location of the concept of almost σ-Dedekind completeness in the Main Inclusion Scheme of Luxemburg and Zaanen [**4**, pp. 137, 174].

THEOREM 2. *Consider the following properties which a Riesz space may possess: super Dedekind completeness (SDC), Dedekind completeness (DC), σ-Dedekind completeness (σDC), almost σ-Dedekind completeness (AσDC), the projection property (PP), the principal projection property (PPP), the property of having sufficiently many projections (SMP), and the property of being Archimedean (Arch). Then the following implications (and only these implications) hold:*

$$\begin{tikzcd}
 & & \sigma\mathrm{DC} \arrow[r] \arrow[ddr] & \mathrm{A}\sigma\mathrm{DC} \arrow[drr] & & \\
\mathrm{SDC} \arrow[r] & \mathrm{DC} \arrow[ur] \arrow[dr] & & & & \mathrm{Arch.} \\
 & & \mathrm{PP} \arrow[r] \arrow[uur] & \mathrm{PPP} \arrow[r] & \mathrm{SMP} \arrow[ur] &
\end{tikzcd}$$

PROOF. The other interrelationships having been shown by Luxemburg and Zaanen, we need only show how almost σ-Dedekind completeness fits into the scheme. As noted before, it is obvious that σ-Dedekind completeness implies almost σ-Dedekind completeness, and that almost σ-Dedekind completeness implies the property of being Archimedean.

(i) *The projection property implies almost σ-Dedekind completeness.* Suppose that L is a Riesz space with the projection property. Let $L^*=L^\delta$ be the Dedekind completion of L; we shall show that L is super order dense in L^*. Let $\theta<u^*\in L^*$ be arbitrary and find an element $u\in L$ such that $u\geqq u^*$. Using a result of Kutty and Quinn [**3**, Theorem 2.6(iii), p. 309], we infer that the components of u in L^* must lie in L, so from Freudenthal's Spectral Theorem [**4**, Theorem 40.2, p. 257], it follows that u^* is a σ-upper element of L and we are done. (See also Lemma 3, p. 895 of [**6**] and Corollary 8.9 of [**5**].) Note that in this case, $L^\sigma=L^\delta$.

(ii) EXAMPLE. *An almost σ-Dedekind complete Riesz space with sufficiently many projections but without the principal projection property.* Let L be the Riesz space of all convergent real sequences with the component-wise ordering. The space L is almost σ-Dedekind complete since it is a super order dense Riesz subspace of the space of bounded real sequences, l_∞, which is σ-Dedekind complete (in fact, super Dedekind complete). But L does not have the principal projection property, since if $u=(1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \cdots)$ then the band generated by u is $\{\{v_n\}\in L: v_n=0$ for n even$\}$ which is not a projection band since $e=(1, 1, 1, \cdots)$ has no decomposition with respect to this band.

(iii) EXAMPLE. *An almost σ-Dedekind complete Riesz space with the principal projection property which is not σ-Dedekind complete and which lacks the projection property.* Let L be the Riesz space of all ultimately constant real sequences with the component-wise ordering. The space L is a super order dense Riesz subspace of l_∞ and it is well known that L satisfies the principal projection property, but is not σ-Dedekind complete and fails to have the projection property [**4**, p. 140].

(iv) EXAMPLE. *A Riesz space with the principal projection property which is not almost σ-Dedekind complete.* Let L denote the vector space of all real-valued functions f of a real variable with the property that there exists a two-tailed sequence $\{x_n\}_{-\infty}^{\infty}$ of real numbers with $\cdots<x_{-1}<x_0<x_1<\cdots$, such that $x_n\to\infty$ as $n\to\infty$ and $x_n\to-\infty$ as $n\to-\infty$ and such that f is constant on the open intervals (x_k, x_{k+1}) for all integers k. Note that L is a Riesz space under the pointwise ordering, and that order convergence in L implies pointwise convergence. The space L does not have the projection property, for if $B=\{u\in L: u(x)=0$ for all irrational $x\}$, then

B is not a projection band. It is clear, though, that L does have the principal projection property for if $v \in L$ is arbitrary, then the band B_v generated by v is simply $B_v=\{f \in L: f(x)=0$ whenever $v(x)=0\}$; by the restrictions on functions in L, this will always be a projection band.

We show next that L is not almost σ-Dedekind complete. Suppose, to the contrary, that L^* is a σ-Dedekind complete Riesz space containing L as a super order dense Riesz subspace. As before, we assume without loss of generality that every positive element of L^* is both a σ-upper and a σ-lower element of L^+. Let $r_1, r_2, \cdots$ be an enumeration of the rationals and let u_n be the characteristic function of the set $\{r_1, r_2, \cdots, r_n\}$. Then $\theta \leqq u_n \uparrow \leqq e$ in L, where e denotes the constant function $e(x)=1$. Since L^* is σ-Dedekind complete, there exists an element $u^* \in L^*$ such that $u_n \uparrow u^* \leqq e$ in L^*. Choose a sequence $\{v_n\}$ of elements of L such that $v_n \downarrow u^*$ in L^*; there is no loss in generality in assuming that $v_n \leqq e$ for all n. It follows easily that $v_n(r)=1$ for all n and all rationals r and that $v_n - u_n \downarrow \theta$ in L. Note that each v_n being an element of L has only countably many discontinuities, so that there exists an irrational number x_0 at which every v_n is continuous. But $u_n(x_0)=0$ for all n so that $v_n(x_0)= v_n(x_0)-u_n(x_0) \downarrow 0$ in $\boldsymbol{R}$, since order convergence implies pointwise convergence. In particular, for some sufficiently large index N we have $v_N(x_0)<\frac{1}{2}$. But v_N is continuous at x_0 by assumption and $v_N(r)=1$ for all rationals r. This is an impossibility so that L is not almost σ-Dedekind complete.

(v) EXAMPLE. *An almost σ-Dedekind complete Riesz space without sufficiently many projections.* Let $L=C[0, 1]$, the Riesz space of continuous real-valued functions on the unit interval with the pointwise ordering. It is well known that L does not have sufficiently many projections [**4**, Example (v), p. 140]. But L is order separable [**4**, Exercise 29.6, p. 170] and is therefore a super order dense Riesz subspace of its Dedekind completion, so that L is almost σ-Dedekind complete. (Cf. Example 8.10 of [**5**].)

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