PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

ADJOINT FUNCTORS INDUCED BY ADJOINT LINEAR TRANSFORMATIONS

PAUL H. PALMQUIST

ABSTRACT. Adjoint linear transformations between Hilbert spaces or, more generally, between dual systems of topological vector spaces induce contravariant functors which are adjoint on the right—essentially a Galois connection between the posets of subsets (or subspaces) of the spaces. Modulo scalars the passage from linear maps to functors is one-to-one; indeed, modulo scalars, two linear transformations are adjoint (hence both are weak continuous) if and only if the induced functors are adjoint.

In a symposium honoring Marshall Stone, Saunders Mac Lane explored a formal analogy of adjoint linear operators on a Hilbert space to adjoint functors between categories [3, pp. 230–232], [4, p. 103]. In contrast, this note singles out a more direct connection: adjoint operators yield adjoint functors.

First, we state the theorem and corollary for Hilbert spaces. Second, as suggested by the referee, we generalize the theorem to dual systems [5, pp. 123, 128] of topological vector spaces and (weak continuous) linear transformations. §3 contains remarks and definitions. §4 proves the general theorem.

1. **Adjoints in Hilbert space.** Consider two complex Hilbert spaces H_0 and H_1, and two maps (=linear transformations) $T_0: H_0 \to H_1$ and $T_1: H_1 \to H_0$. (For notation, let $i, j = 0$ or 1, and $i \neq j$ in the sequel.) The inner product on H_i is denoted by $(\, , \,)_i$. Let $\mathscr{C}_i$ be the (small) category whose set of objects $|\mathscr{C}_i|$ consists of all subsets of H_i and whose only arrows are inclusions ($X \to Y$ in $\mathscr{C}_i$ iff $X \subseteq Y \subseteq H_i$). For A in $|\mathscr{C}_i|$, let

$$D_i A = \{b \in H_j \mid (T_i a, b)_j = 0, \forall a \in A\}.$$

THEOREM 1. *With T_i and D_i as above the following statements are true.*

(i) *This yields a contravariant functor D_i from $\mathscr{C}_i$ to $\mathscr{C}_j$.*

Received by the editors January 2, 1973 and, in revised form, September 17, 1973.

AMS (MOS) subject classifications (1970). Primary 18A40, 47A05; Secondary 06A15, 46C10, 46A20.

Key words and phrases. Adjoint, adjoint linear operators, adjoint functors, adjoint on the right, Galois connection, Hilbert space, poset, topological vector space, dual system.

(ii) *If the maps T_0 and T_1 are adjoint, then the induced contravariant functors D_0 and D_1 are adjoint on the right.*

(iii) *If the induced contravariant functors D_0 and D_1 are adjoint on the right, then there is a scalar λ such that λT_0 and T_1 are adjoint. (If $T_0 \neq 0$, then λ is unique.)*

(iv) *Modulo scalars the passage from maps to functors is one-to-one.*

We state an immediate consequence.

COROLLARY. *If T_0 is selfadjoint, then D_0 is selfadjoint.*

2. **Adjoints in dual systems.** Consider two dual systems E_i, F_i, $(\ ,\ )_i$, each consisting of a pair of vector spaces equipped with a nonsingular bilinear form over a (nondiscrete valuated) field K. Consider two linear transformations $T_0: E_0 \to E_1$ and $T_1: F_1 \to F_0$. Let $\mathscr{C}_0$ ($\mathscr{C}_1$) be the small category whose set of objects $|\mathscr{C}_0|$ ($|\mathscr{C}_1|$) consists of all subsets of E_0 (F_1), and whose only arrows are inclusions, for example $X \to Y$ in $\mathscr{C}_0$ iff $X \subseteq Y \subseteq E_0$. For $A \in |\mathscr{C}_0|$ and $B \in |\mathscr{C}_1|$, let

$$D_0 A = \{y \in F_1 \mid (T_0 a, y)_1 = 0, \forall a \in A\},$$

and

$$D_1 B = \{x \in E_0 \mid (x, T_1 b)_0 = 0, \forall b \in B\}.$$

THEOREM 2. *For dual systems, with T_i and D_i as above, the statements of Theorem 1 hold.*

3. **Definitions, remarks, and examples.** Given a complex Hilbert space H_i, define a dual system by $E_i = H_i$ and F_i equals H_i with the conjugate complex scalar multiplication thereby allowing the bilinear form on $E_i \times F_i$ to be the given Hermitian inner product on H_i. Note that a function $T_1: H_1 \to H_0$ is linear iff it is linear as $F_1 \to F_0$. Thus the functors D_i of §2 generalize those of §1.

Recall that the pair of maps T_0 and T_1 between Hilbert spaces (dual systems) are *adjoint* [**2**, p. 39] ([**5**, p. 128]) if for all (x, y) in $H_0 \times H_1$ (in $E_0 \times F_1$)

$$\text{(AM)} \qquad (T_0 x, y)_1 = (x, T_1 y)_0,$$

while two contravariant functors D_0 and D_1 are *adjoint on the right* [**1**, p. 81] if we have the natural isomorphism of hom-sets

$$\text{(AF)} \qquad \mathscr{C}_1(Y, D_0 X) \simeq \mathscr{C}_0(X, D_1 Y), \qquad \forall (X, Y) \in |\mathscr{C}_0| \times |\mathscr{C}_1|.$$

For Hilbert spaces these are both symmetric relationships (unlike the adjointness of covariant functors). In the special case where $T_0 = T_1 =$ the identity map on H_0, we have $D_0 A = A^{\perp}$ the *orthogonal complement* of the subset A.

Since the categories $\mathscr{C}_0$ and $\mathscr{C}_1$ are essentially posets, having D_0 and D_1 adjoint on the right is essentially a *Galois connection* [**4**, Theorem 1, p. 93].

All statements and proofs in this note hold if the categories $\mathscr{C}_i$ are replaced by their full subcategories whose objects are closed subspaces and if the singleton subset $\{x\}$ is replaced by its span $\{\lambda x | \lambda \in K\}$.

For the linear transformation $T_0: E_0 \to E_1$, existence of an adjoint is equivalent to *weak continuity* [**5**, p. 128].

Finally, everything in Theorem 1 is a special case of Theorem 2. So it only remains to prove the latter.

4. **Proof of the general theorem.** (i) Given the arrow $X \to Y$ in $\mathscr{C}_0$, one verifies $D_0 Y \subseteq D_0 X \subseteq F_1$; similarly, D_1 is functorial.

(ii) If the maps T_0 and T_1 are adjoint, then the equation (AM) holds. We must show (AF) holds. This follows from a reversible chain of implications: $Y \subseteq D_0 X$ implies $\forall (x, y) \in X \times Y$ that $(T_0 x, y)_1 = 0$, which by adjointness (AM) implies $\forall (x, y) \in X \times Y$ that $(x, T_1 y)_0 = 0$, hence $X \subseteq D_1 Y$.

(iii) Given (AF), the left side of (AM) equals 0 iff the right side of (AM) equals 0, since $\{y\} \subseteq D_0\{x\}$ iff $\{x\} \subseteq D_1\{y\}$. Thus $T_0 = 0$ implies (AM) holds, since both sides are 0.

Otherwise pick a vector $x \in E_0$ with $T_0 x \neq 0$, and consider the linear functionals $f = (T_0 x, \ )_1$ and $g = (x, T_1)_0 : F_1 \to K$. For $y \in F_1$, $fy = 0$ implies $gy = 0$ by (AF). Thus there is a unique scalar $\lambda \in K$ with $g = \lambda f$. It remains to show that

$$(\lambda T_0 a, b) = (a, T_1 b), \forall (a, b) \in E_0 \times F_1.$$

The equation holds if $T_0 a$ is a multiple of $T_0 x$, since $T_0 a = \mu T_0 x$ implies $a - \mu x = n \in \ker T_0$, which implies $(n, T_1 b)_0 = 0$ by (AF). Otherwise, if $T_0 a$ and $T_0 x$ are linearly independent, denote $x' = a$ and $x'' = x' + x$ with corresponding linear functionals and scalars as before satisfying $g' = \lambda' f'$ and $g'' = \lambda'' f''$. Consider the chain of equalities:

$$\lambda' f' + \lambda f = g' + g = g'' = \lambda'' f'' = \lambda'' f' + \lambda'' f.$$

Hence we have $\lambda' = \lambda'' = \lambda$, thus the adjointness equation holds for all $a \in E_0$. *Note.* We have freely used the bilinearity of the bilinear forms and the linearity of the maps, but no continuity or prior existence of adjoints was assumed.

(iv) If the two maps T_0 and T_0' induce the same functor D_0, we will show $T_0' = \lambda T_0$ for some scalar. We may assume T_0' has an adjoint T_1 by enlarging F_0 and F_1 if necessary [**5**, p. 128]. Then by (iii), there is $\lambda \in K$ such that λT_0 is adjoint to T_1. Hence T_0' equals λT_0.

REFERENCES

1. P. Freyd, *Abelian categories: An introduction to the theory of functors*, Harper and Row, New York, 1964.

2. P. Halmos, *Introduction to Hilbert spaces*, 2nd ed., Chelsea, New York, 1957.

3. S. Mac Lane, *The influence of M. H. Stone on the origins of category theory*, Conference on Functional Analysis and Related Fields, F. E. Browder, Ed., Springer, New York, 1970.

4. ———, *Categories for the working mathematician*, Springer, New York, 1971.

5. H. H. Schaefer, *Topological vector spaces*, Macmillan, New York, 1966. MR 33 #1689.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92664