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FIXED POINTS OF SEVERAL CLASSES OF NONLINEAR MAPPINGS IN BANACH SPACE

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ABSTRACT. In the first part of the paper conditions for the existence of ordinary and higher order fixed points of individual and commutative families of nonlinear operators are obtained.

The second part deals with the existence of fixed points of an operator $T: C \to X$ whose graph is closed in the Cartesian product topology induced by the strong topology in C and the weak topology in X.

The convergence to fixed points of sequences of successive approximations is considered in both parts.

1. **Introduction.** The classical fixed-point theorem of Schauder states that if $T: C \to C$ is a continuous mapping of a closed convex subset of a Banach space X into itself and if $T(C)$ is contained in a compact subset of C, then there exists an $x \in C$ such that $Tx = x$. This result has been extended by Kirk [5] and others.

The purpose of this paper is to obtain additional conditions for the existence of fixed points, including fixed points of higher order. Common fixed points of families of operators and the convergence of sequences of successive approximations are also considered.

The following definitions will be required:

DEFINITION 1. Let X be a Banach space and K a set in X. A mapping $T: K \to X$ is called *nonexpansive* if $\|Tx - Ty\| \leqq \|x - y\|$ for all x and y in K.

DEFINITION 2. A Banach space is called *strictly convex* if for any pair of elements $x, y \in X$ from $\|x+y\| = \|x\| + \|y\|$ it follows that $x = \lambda y$, $\lambda > 0$ (or, in the trivial case, $y = 0$).

DEFINITION 3. If T is a mapping on a Banach space, a point satisfying the condition $T^n x = x$ for some positive integer n is called a *fixed point of order n*.

DEFINITION 4. A family $\{T_\lambda\}_{\lambda \in \Lambda}$ of mappings of a set K into itself is called *commutative* if $T_\lambda T_\mu = T_\mu T_\lambda$ for all λ, μ in Λ.

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2. **Fixed points of order n.** Let T be nonexpansive. Since

$$\|T^n x - T^n y\| = \|T(T^{n-1}x) - T(T^{n-1}y)\| \leqq \|T^{n-1}x - T^{n-1}y\|,$$

it follows that $\|T^n x - T^n y\| \leqq \|x-y\|$. As a consequence, higher order fixed points possess many properties of ordinary fixed points. For example, following the procedure for ordinary fixed points, one can show:

PROPOSITION 1. *If C is a closed convex subset of a strictly convex Banach space X, then for every nonexpansive mapping $T_\lambda: C \to X$, the set $F_\lambda^n = \{x \in C: T_\lambda^n x = x\}$ is convex and closed.*

PROOF. If $x, y \in F_\lambda^n$, then for z on the segment joining x and y, we have

$$\begin{aligned} \|x - y\| &\leqq \|x - T_\lambda^n z\| + \|T_\lambda^n z - y\| \\ &= \|T_\lambda^n x - T_\lambda^n z\| + \|T_\lambda^n z - T_\lambda^n y\| \\ &\leqq \|x - z\| + \|z - y\| = \|x - y\|. \end{aligned}$$

Hence $\|x-y\| = \|x - T_\lambda^n z\| + \|T_\lambda^n z - y\|$, and consequently $T_\lambda^n z$ lies on the segment joining x and y, by strict convexity. Since T_λ^n is nonexpansive, $\|x - T_\lambda^n z\| = \|T_\lambda^n x - T_\lambda^n z\| \leqq \|x-z\|$, so that $T_\lambda^n z$ lies on the segment joining x and z. Similarly, $T_\lambda^n z$ lies on the segment joining z and y, which is possible only if $T_\lambda^n z = z$. That F_λ^n is closed follows directly from the continuity of T_λ.

The mapping $U = \frac{1}{2}(I+T)$ is nonexpansive and has the same fixed points as T. The following theorem states conditions for the convergence of the sequence $\{U^n x\}$ to a fixed point of U:

THEOREM A (EDELSTEIN [4]). *Let K be a closed convex subset of a strictly convex Banach space X, $T: K \to K$ a nonexpansive mapping, and suppose $T(K)$ is contained in a compact subset K_1 of K. Then the sequence $\{U^n x\}$ converges to a fixed point of T for all $x \in K$.*

Suppose $\{T_\lambda\}_{\lambda \in \Lambda}$ is a commutative family each member of which satisfies the conditions of the theorem. If the set of fixed points of T_λ is denoted by F_λ, then $F_\lambda \neq \varnothing$ for all λ by the theorem. Under additional conditions on $\{F_\lambda\}$, the family $\{T_\lambda\}$ has a common fixed point.

THEOREM 1. *Let K be a closed convex subset of a strictly convex Banach space X and $\{T_\lambda\}_{\lambda \in \Lambda}$ be a commutative family of mappings each member of which satisfies the conditions of the above theorem. Suppose for every F_α ($\alpha \in \Lambda$) having a positive diameter there exists $\lambda \in \Lambda$ such that F_λ is a proper subset of F_α. Then $\{T_\lambda\}$ has a common fixed point in K.*

PROOF. Let $x \in F_\lambda$. Then $\lim_{n \to \infty} U_\mu^n x = x_0$, where $U_\mu x_0 = x_0$ by the theorem above. Since x is a fixed point of U_λ (and hence of T_λ),

$\lim_{n\to\infty} U_\lambda(U_\mu^n x) = \lim_{n\to\infty} U_\mu^n(U_\lambda x) = x_0$, so that by the continuity of U_λ, $U_\lambda x_0 = x_0$. Consequently, any two members of the family $\{T_\lambda\}_{\lambda\in\Lambda}$ have a fixed point in common.

Consider any decreasing chain of sets $\{F_\alpha\}$. Since the sets are closed by Proposition 1 and possess the finite intersection property, the family has a nonempty intersection by the compactness of the range of each T_λ.

Hence, by the Kuratowski-Zorn lemma, there exists a minimal element F_λ. If the diameter of F_λ is positive, there exists $\alpha \in \Lambda$ such that F_α is a proper subset of F_λ, contradicting the minimality. Therefore F_λ has exactly one element x_1, which is the unique fixed point of T_λ.

Since any two mappings have a fixed point in common, x_1 is a common fixed point of $\{T_\lambda\}_{\lambda\in\Lambda}$.

The following theorem gives conditions for the convergence of $\{T^n x\}$ to a fixed point of T:

THEOREM B (BROWDER AND PETRYSHYN [2]). *Let C be a closed subset of a Banach space X. If the mapping $T: C \to C$ is nonexpansive,*

$$\lim_{n\to\infty} \|T^{n+1}x - T^n x\| = 0$$

for all $x \in C$, and the mapping $I - T$ maps closed bounded subsets of C into bounded subsets of X and if the set of fixed points of T in C is nonempty, then for any x in C the sequence $\{T^n x\}$ is convergent to a fixed point of T.

Using this result the proof of Theorem 1 carries over into

THEOREM 2. *Let C be a compact subset of a Banach space X. If the members of the family $\{T_\lambda\}_{\lambda\in\Lambda}$ of mappings satisfy the hypothesis of Theorem* B, *and if for every F_α $(\alpha \in \Lambda)$ having a positive diameter there exists $\lambda \in \Lambda$ such that F_λ is a proper subset of F_α, then $\{T_\lambda\}$ has a common fixed point in C.*

Using the notion of a fixed point of order n, we get

THEOREM 3. *Let C be a closed subset of a Banach space, and $\{T_\lambda\}_{\lambda\in\Lambda}$ a commutative family of continuous functions from C into itself. Suppose for every $x \in C$, $\{T_\lambda^n x\}$ converges to a fixed point of order m of T_λ for every $\lambda \in \Lambda$. Then $\{T_\lambda\}$ has a common fixed point of order m in C.*

PROOF. Let $x \in F_\lambda^m$. Then $\lim_{n\to\infty} T_\mu^n x = x_0$, where $T_\mu^m x_0 = x_0$, and $\lim_{n\to\infty} T_\lambda^m(T_\mu^n x) = \lim_{n\to\infty} T_\mu^n(T_\lambda^m x) = x_0$. Hence, by continuity, $T_\lambda^m x_0 = x_0$, and $x_0 \in F_\lambda^m \cap F_\mu^m$.

Suppose Λ has been well-ordered and $\{\lambda_1, \lambda_2, \cdots, \lambda_\omega, \cdots, \lambda_\alpha, \cdots\}$ is a transfinite sequence from Λ. Suppose further that $F^m = F_{\lambda_1}^m \cap F_{\lambda_2}^m \cap \cdots \cap F_{\lambda_\beta}^m$

for all $\beta<\alpha$ is nonempty. Consider T_{λ_α} and $x\in F^m$; by hypothesis, $\{T^n_{\lambda_\alpha}x\}$ converges to a fixed point x_2 of order m of T_{λ_α}, i.e., $x_2\in F^m_{\lambda_\alpha}$. But $x_2\in F^m$ by the first part, so that $F^m\cap F^m_{\lambda_\alpha}$ is also nonempty.

THEOREM 4. *Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be a commutative family of operators from a subset C of a Banach space into itself, and let x_0 be a common fixed point. Suppose for every $y\in C$, there exists $\lambda\in\Lambda$ such that* $\lim_{n\to\infty} T^n_\lambda y=x_0$. *Let $T:C\to C$ be a continuous operator commuting with $\{T_\lambda\}$; then*

(a) *if x is an rth order fixed point of T (for some $x\in C$), x_0 is also an rth order fixed point of T;*

(b) *if* $\lim_{n\to\infty} T^n_\lambda x=x_0$ *for all λ and some $x\in C$, x_0 is an ordinary fixed point of T.*

PROOF. (a) Consider T^nx, $n=1,2,\cdots$. By hypothesis,

$$\lim_{m\to\infty} T^m_{\lambda_n}(T^nx) = x_0, \qquad \lambda_1,\lambda_2,\cdots\in\Lambda,$$

and by commutativity,

$$\lim_{m\to\infty} T^n(T^m_{\lambda_n}x) = x_0, \qquad \lambda_1,\lambda_2,\cdots\in\Lambda.$$

Now if x is an rth order fixed point of T, $T^rx=x$ and $\lim_{m\to\infty} T^m_{\lambda_r}(T^rx)=x_0$ (i.e., for $n=r$, $\lim_{m\to\infty} T^m_{\lambda_n}x=x_0$). Hence by continuity $T^rx_0=x_0$.

(b) If $\lim_{n\to\infty} T^n_\lambda x=x_0$ for all λ, $T^nx_0=x_0$ for all n, and x_0 is an ordinary fixed point of T.

Note. If C is a bounded closed set with "normal structure" (i.e., there exists a point in C which is not diametral) of a reflexive and strictly convex Banach space, and if the mappings are nonexpansive, the assumption that $\{T_\lambda\}$ has a common fixed point follows and may be omitted (Browder [1]). Similarly, the assumption may be omitted if the mappings are nonexpansive and C is compact and convex (De Marr [3]).

Returning to the question of computing the common fixed point, one can obtain a condition in terms of higher order fixed points.

THEOREM 5. *Let $\{T_\lambda\}_{\lambda\in\Lambda}$ be a commutative family of operators on a subset C of a Banach space into itself, and let x_0 be a common fixed point. Suppose that for every $x\in C$ there exists a $\lambda\in\Lambda$ such that* $\lim_{m\to\infty} T^m_\lambda x=x_0$, *and suppose further that $\{T^m_\lambda x\}$ converges for all λ. Let $T:C\to C$ be a continuous one-to-one operator commuting with the family $\{T_\lambda\}$ and such that $T^rx_0=x_0$ for some r. Then there exists $\alpha\in\Lambda$ such that* $\lim_{m\to\infty} T^m_\alpha x=x_0$ *for every $x\in C$.*

PROOF. Consider T^nx, $n=1,2,\cdots$. By hypothesis

$$\lim_{m\to\infty} T^m_{\lambda_n}(T^nx) = x_0, \qquad \lambda_1,\lambda_2,\cdots\in\Lambda,$$

and

$$\lim_{m\to\infty} T^n(T^m_{\lambda_n}x) = x_0, \qquad \lambda_1, \lambda_2, \cdots \in \Lambda.$$

Since $\{T^m_{\lambda_n}x\}$ converges and T is one-to-one, while $T^r x_0 = x_0$ for some r, $\lim_{m\to\infty} T^m_{\lambda_r}x = x_0$ for all $x \in C$ and some $\lambda_r \in \Lambda$.

More interesting consequences are:

COROLLARY 1. *If x_0 is an ordinary fixed point of T, there exists a countably infinite subfamily $\{T_{\lambda_n}\}$ of $\{T_\lambda\}_{\lambda\in\Lambda}$ such that for every $x \in C$, $\lim_{m\to\infty} T^m_{\lambda_n}x = x_0$, $\lambda_1, \lambda_2, \cdots \in \Lambda$ (for then $T^n x_0 = x_0$ for all n).*

COROLLARY 2. *If T_λ is continuous and one-to-one for some $\lambda \in \Lambda$, the conclusion of Corollary 1 also follows.*

3. **Fixed points of HH_w-closed mappings.** The following definition is due to Browder and Petryshyn [2]:

DEFINITION 5. Let H be a subset of a Banach space. A mapping $T: H \to H$ will be called *H_wH-closed* (*demiclosed* in the terminology of [2]) if its graph in $H \times H$ is closed in the Cartesian product topology induced in $H \times H$ by the weak topology in H (denoted by H_w) and the strong topology in H.

In analogous fashion we shall define HH_w-closed:

DEFINITION 6. Let H be a subset of a Banach space. A mapping $T: H \to H$ will be called *HH_w-closed* if for any sequence $\{x_n\} \subset H$ which converges strongly to x in H the weak convergence of the sequence $\{Tx_n\}$ to y in H implies that $Tx = y$.

THEOREM 6. *Let H be a closed subset of a Banach space and T a mapping from H to H. Assume that $I-T$ is HH_w-closed and $(I-T)^{-1}$ exists and is continuous from the weak topology in $R(I-T)$ to the strong topology in H. Assume further that there exists a sequence $\{x_n\} \subset H$ such that $\{(I-T)x_n\}$ is weakly convergent to 0 as $n \to \infty$. Then T has a fixed point in H.*

PROOF. Let $\{x_n\}$ be the sequence for which $\{(I-T)x_n\}$ converges weakly to 0. Since $(I-T)^{-1}$ is continuous from the weak topology in $R(I-T)$ to the strong topology in H, $\{x_n\}$ converges strongly to some x_0 as $n \to \infty$, where $x_0 \in H$, since H is closed. Because $I-T$ is HH_w-closed, $(I-T)x_0 = 0$, so that $Tx_0 = x_0$.

Another variation of this theorem is

THEOREM 7. *Let H be a closed subset of a Banach space and T a mapping from H to H. Assume that $I-T$ is HH_w-closed and $(I-T)^{-1}$ exists and is continuous from the weak topology in $R(I-T)$ to the strong* *topology in H. Finally, assume that for every $x \in H$ the sequence $\{T^n x - T^{n+1}x\}$ is weakly convergent to 0 as $n \to \infty$. Then T has a fixed point in H.*

PROOF. Consider the sequence $\{T^n x\}$. By hypothesis $\{(T^n - T^{n+1})x\} = \{(I-T)T^n x\}$ converges weakly to 0 as $n \to \infty$. Since $(I-T)^{-1}$ is continuous in the above sense and $I-T$ is HH_w-closed, $\{T^n x\}$ converges strongly to some $x_0 \in H$, and $(I-T)x_0 = 0$.

Since $\{T^n x\}$ may be considered a sequence of successive approximations, we obtain from the proof of Theorem 7 the following

COROLLARY 3. *For every $x \in H$ the sequence $\{T^n x\}$ of successive approximations converges to a fixed point of H.*

THEOREM 8. *Let H be a compact subset of a Banach space, and let $\{T_\alpha\}_{\alpha \in \Lambda}$ be a commutative family of mappings from H to H. For each $\alpha \in \Lambda$ assume that $I - T_\alpha$ is HH_w-closed and $(I-T_\alpha)^{-1}$ continuous from the weak topology in $R(I-T_\alpha)$ to the strong topology in H. Assume further that for every $x \in H$ and $\alpha \in \Lambda$ the sequence $\{(T_\alpha^n - T_\alpha^{n+1})x\}$ converges weakly to 0 as $n \to \infty$. Then $\{T_\alpha\}_{\alpha\in\Lambda}$ has a common fixed point in H.*

PROOF. Let K_α be the set of fixed points of T_α. Then $K_\alpha \neq \varnothing$ for every $\alpha \in \Lambda$ by Theorem 7.

Let $\{x_n\}$ be a sequence in K_α converging strongly to x as $n \to \infty$. Then $\{T_\alpha x_n\}$ converges strongly to x (and hence weakly), so that $\{(I-T_\alpha)x_n\}$ converges weakly to 0 as $n \to \infty$. Since $I - T_\alpha$ is HH_w-closed, $(I-T_\alpha)x = 0$, i.e., $T_\alpha x = x$. Consequently, K_α is closed for every $\alpha \in \Lambda$.

Since H is compact, it is sufficient to show that $\{K_\alpha\}_{\alpha\in\Lambda}$ has the finite intersection property.

Toward this end we first observe that for any $\alpha \in \Lambda$ and $x \in K_\alpha$

$$T_\alpha(T_\lambda x) = T_\lambda(T_\alpha x) = T_\lambda x$$

by commutativity. Hence for any $\alpha \in \Lambda$, T_λ maps K_α into itself.

Proceeding by induction, we choose any sequence $\alpha_1, \alpha_2, \cdots, \alpha_n$ from Λ with $K = K_{\alpha_1} \cap K_{\alpha_2} \cap \cdots \cap K_{\alpha_{n-1}}$ assumed nonempty. Recalling that T_{α_n} maps K_{α_i} $(i = 1, 2, \cdots, n-1)$ into itself, we may consider T_{α_n} to be a mapping from K to K. Since K is closed, $K_{\alpha_n} \neq \varnothing$ by Theorem 7, so that $K \cap K_{\alpha_n}$ is also nonempty.

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