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APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS

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ABSTRACT. A condition is given for nonexpansive mappings which assures convergence of certain iterates to a fixed point of the mapping in a uniformly convex Banach space. A relationship between the given condition and the requirement of demicompactness is established.

Introduction. Browder [**1**] and Kirk [**7**] have shown that a nonexpansive mapping T which maps a closed, bounded, convex subset C of a uniformly convex Banach space into itself has a nonempty fixed point set in C. In general, however, for arbitrary $x \in C$ the Picard iterates $T^n x$ do not converge to a fixed point of T. It will be shown that if T satisfies one additional condition, then an iterative process of the type introduced by W. R. Mann [**8**] converges to a fixed point of T. For nonexpansive mappings T which have fixed points, this additional condition is weaker than the requirement that T be demicompact.

Convergence to a fixed point. Let X be a Banach space with norm $|\cdot|$ and C a convex subset of X. A self-mapping T of C is said to be *nonexpansive* provided $|Tx-Ty| \leqq |x-y|$ holds for all $x, y \in C$. A mapping $T: C \to C$ with nonempty fixed point set F in C will be said to satisfy *Condition* I if there is a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0)=0$, $f(r)>0$ for $r \in (0, \infty)$, such that $|x-Tx| \geqq f(d(x, F))$ for all $x \in C$, where $d(x, F)=\inf\{|x-z| : z \in F\}$.

Let P denote the set of positive integers. For $x_1 \in C$, $M(x_1, t_n, T)$ is the sequence $\{x_n\}$ defined by $x_{n+1}=(1-t_n)x_n+t_n Tx_n$ where $t_n \in [a, b]$ for all $n \in P$ and $0<a<b<1$. This iterative process has been previously investigated by Dotson in [**4**].

Our main result for nonexpansive mappings is the following:

THEOREM 1. *Suppose X is a uniformly convex Banach space, C is a closed, bounded, convex, nonempty subset of X, and T is a nonexpansive mapping of C into C. Let F denote the fixed point set of T in C, and suppose*

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T satisfies Condition I. *Then for any $x_1 \in C$, $M(x_1, t_n, T)$ converges to a member of F.*

Given that F is nonempty (which in Theorem 1 is assured by the Browder-Kirk theorem [**1**], [**7**]) the proof that $M(x_1, t_n, T)$ converges to a fixed point uses only the fact that T is nonexpansive about its fixed points (see Theorem 2 below). Theorem 1 will follow immediately as a corollary of Theorem 2. As in [**3**], a self-mapping T of C will be called *quasi-nonexpansive* provided T has a fixed point in C and if $p \in C$ is a fixed point of T then $|Tx-p| \leqq |x-p|$ is true for all $x \in C$. The class of quasi-nonexpansive mappings includes continuous as well as discontinuous mappings which are not nonexpansive. One can easily prove that $T: C \to C$ is quasi-nonexpansive if T has a fixed point in C and for $x, y \in C$ satisfies either

$$\text{(A)} \qquad |Tx - Ty| \leqq \beta[|x - Tx| + |y - Ty|], \qquad 0 \leqq \beta \leqq 1/2,$$

or

$$\text{(B)} \qquad |Tx - Ty| \leqq a\,|x - Tx| + b\,|y - Ty| + c\,|x - y|,$$

where $a, b, c > 0$ and $a+b+c \leqq 1$.

Mappings which satisfy the requirement (A) or (B) have been recently investigated by Kannan [**6**] and Reich [**12**] respectively.

For a uniformly convex Banach space X, Dotson [**4**] has shown that if $\{w_n\}$ and $\{y_n\}$ are sequences in the closed unit ball of X and if $\{z_n\} = \{(1-t_n)w_n + t_n y_n\}$ satisfies $\lim|z_n| = 1$, where $t_n \in [a, b]$ for $0 < a < b < 1$, then $\lim|w_n - y_n| = 0$. This result will be used in the proof of

THEOREM 2. *Suppose X is a uniformly convex Banach space, C is a closed, convex subset of X and T is a quasi-nonexpansive mapping of C into C. If T satisfies Condition* I, *where F is the fixed point set of T in C, then for arbitrary $x_1 \in C$, $M(x_1, t_n, T)$ converges to a member of F.*

PROOF. If $x_1 \in F$ the result is trivial, so we assume $x_1 \in C \sim F$. For arbitrary $z \in F$ we have for $n \in P$ that $|Tx_n - z| \leqq |x_n - z|$ and so

$$|x_{n+1} - z| \leqq (1 - t_n)\,|x_n - z| + t_n\,|Tx_n - z| \leqq |x_n - z|.$$

Thus, $d(x_{n+1}, F) \leqq d(x_n, F)$ for all $n \in P$. The sequence $\{d(x_n, F)\}$ is nonincreasing and bounded below, so $\lim d(x_n, F)$ exists. We now show (indirectly) that this limit must be zero, and in turn, that $\{x_n\}$ converges to a member of F.

Suppose $\lim d(x_n, F) = b > 0$. Then for $z_0 \in F$, $\lim|x_n - z_0| = b' \geqq b > 0$. Choose $N > 0$ such that $|x_n - z_0| \leqq 2b'$ for $n \geqq N$. If we let $y_n = (Tx_n - z_0)/|x_n - z_0|$ and $w_n = (x_n - z_0)/|x_n - z_0|$, then $|y_n| \leqq 1$ and $|w_n| = 1$

for all $n \in P$; and for $n \geqq N$

$$|w_n - y_n| = \frac{|x_n - Tx_n|}{|x_n - z_0|} \geqq \frac{f(d(x_n, F))}{|x_n - z_0|} \geqq \frac{f(b)}{2b'} > 0.$$

Therefore, $\lim|w_n - y_n| \neq 0$. Moreover

$$\lim |(1 - t_n)w_n + t_n y_n| = \lim |x_{n+1} - z_0|/|x_n - z_0| = b'/b' = 1.$$

However, by the contrapositive of Dotson's result [**4**] stated above, since $\lim|w_n - y_n| \neq 0$ then the existence of $\lim|(1-t_n)w_n + t_n y_n|$ implies $\lim|(1-t_n)w_n + t_n y_n| \neq 1$, a contradiction. Therefore, $\lim d(x_n, F) = 0$. We show that this implies $\{x_n\}$ converges to an element of F.

Since $\lim d(x_n, F) = 0$, given $\varepsilon > 0$ there exists $N_\varepsilon > 0$ and $z_\varepsilon \in F$ such that $|x_{N_\varepsilon} - z_\varepsilon| < \varepsilon$, which implies $|x_n - z_\varepsilon| < \varepsilon$ for all $n \geqq N_\varepsilon$. Thus, if $\varepsilon_k = 1/2^k$ for $k \in P$, then corresponding to each ε_k there is an $N_k > 0$ and a $z_k \in F$ such that $|x_n - z_k| \leqq \varepsilon_k/4$ for all $n \geqq N_k$. We require $N_{k+1} \geqq N_k$ for all $k \in P$. We have for all $k \in P$

$$|z_k - z_{k+1}| = |z_k - x_{N_{k+1}} + x_{N_{k+1}} - z_{k+1}| < \varepsilon_k/4 + \varepsilon_{k+1}/4 = 3\varepsilon_{k+1}/4.$$

Let $S(z, \varepsilon) = \{x \in X : |x - z| \leqq \varepsilon\}$ denote the closed sphere centered at z of radius ε. For $x \in S(z_{k+1}, \varepsilon_{k+1})$ we have

$$|z_k - x| = |z_k - z_{k+1} + z_{k+1} - x| < 3\varepsilon_{k+1}/4 + \varepsilon_{k+1} < 2\varepsilon_{k+1} = \varepsilon_k.$$

That is, $S(z_{k+1}, \varepsilon_{k+1}) \subseteq S(z_k, \varepsilon_k)$ for $k \in P$. Thus, $\{S(z_k, \varepsilon_k)\}$ is a nested sequence of nonvoid closed spheres with radii ε_k tending to zero. By the Cantor intersection theorem, $\bigcap_{k \in P} S(z_k, \varepsilon_k)$ contains exactly one point, say w. The fixed point set F is closed by [**3**] and the sequence $\{z_k\}$ from F converges to w, so $w \in F$. Since $|x_n - z_k| < \varepsilon_k/4$ for $n \geqq N_k$, we have $\{x_n\} \to w$. Q.E.D.

Note that in Theorem 2 the set C is not required to be bounded; however, boundedness of C is needed in Theorem 1 to apply the Browder-Kirk theorem.

In the preceding theorems, the fixed point of T to which $M(x_1, t_n, T)$ converges depends, in general, on the initial approximation x_1 as well as the values of the t_n. Also, $M(x_1, t_n, T)$ need not converge to the fixed point of T nearest x_1. The following example can be used to verify each of these facts. Let X be the space R^2 with the Euclidean norm and, with (r, θ) denoting polar coordinates, let $C = \{(r, \theta) : 0 \leqq r \leqq 1, -\pi/2 \leqq \theta \leqq -\pi/4\}$. Define $T: C \to C$ by $T[(r, \theta)] = (r, -\pi/2)$ for each point (r, θ) in C. The set of fixed points of T is the line segment $F = \{(r, -\pi/2) : 0 \leqq r \leqq 1\}$.

On Condition I. If $T:C\to C$ has a nonvoid fixed point set F, then T will be said to satisfy Condition II provided there exists a real number $\alpha>0$ such that $|x-Tx|\geqq\alpha\cdot d(x,F)$ holds for all $x\in C$, where as before $d(x,F)=\inf_{z\in F}|x-z|$. Clearly mappings which satisfy Condition II also satisfy Condition I, and in some cases Condition II is easily verified. In the example above, Condition II holds with $\alpha=1$. If T rotates points of the unit ball of R^2 through an angle $\pi/2$, then Condition II holds with $\alpha=\sqrt{2}$.

Condition II is similar to, but less restrictive than, a requirement imposed by Outlaw in [**10**, Theorem 2]. Mappings satisfying Outlaw's condition can have at most a single fixed point; his second theorem follows as a special case of Theorem 2.

If $T:C\to C$ satisfies either requirement (A) or (B) (see above) and has a fixed point in C, then it is easily shown that T has a unique fixed point [**6**], [**12**]. In [**6**, Theorem 2] Kannan proves under certain conditions that for $x_1\in C$, $M(x_1,\frac{1}{2},T)$ converges to the fixed point of T if T satisfies (A). We extend his result with

THEOREM 3. *Let C be a subset of a Banach space X and T a mapping of C into C which satisfies either* (A) *or* (B) *and has a* (*unique*) *fixed point in C. Then T satisfies Condition* II. *If C is closed and convex and X is uniformly convex then for any $x_1\in C$, $M(x_1,t_n,T)$ converges to the fixed point of T.*

PROOF. Assume T satisfies requirement (B) and let p be the unique fixed point of T. Then for $x\in C$

$$|Tx-p|=|Tx-Tp|\leqq a\,|x-Tx|+c\,|x-p|$$

and

$$|Tx-p|\geqq\big|\,|Tx-x|-|x-p|\,\big|\geqq|x-p|-|x-Tx|.$$

Hence

$$a\,|x-Tx|+c\,|x-p|\geqq|x-p|-|x-Tx|,$$

so $|x-Tx|\geqq[(1-c)/(1+a)]|x-p|$. The constant $(1-c)/(1+a)$ is positive since $0<a,c<1$. Thus Condition II holds. A similar argument applies if T is a mapping of the type (A).

Since T is quasi-nonexpansive and satisfies Condition I, the second assertion of the theorem follows directly from Theorem 2. Q.E.D.

We now establish a relationship between mappings which satisfy Condition I and those which are demicompact, beginning with

LEMMA 1. *Suppose C is a closed, bounded subset of a Banach space X and $T:C\to C$ has a nonempty fixed point set F in C. If $I-T$ maps closed bounded subsets of C onto closed subsets of X, then T satisfies Condition* I *on C.*

PROOF. Let $M=\sup\{d(x, F):x \in C\}$. If $M=0$ then $F=C$ and Condition I follows trivially. Suppose $M>0$; for $0<r<M$ define $C_r=\{x \in C: d(x, F)\geqq r\}$ and $f(r)=\inf\{|x-Tx|:x \in C_r\}$. Note that each set C_r is nonempty, closed and bounded. We prove that for arbitrary r, $0<r<M$, $f(r)>0$.

By hypothesis, $(I-T)C_r=\{x-Tx:x \in C_r\}$ is closed. If $\theta \in (I-T)C_r$ then $\theta=z-Tz$ for some $z \in C_r$, which implies $z=Tz$ and thus $z \in F$. But $d(z, F)\geqq r>0$, a contradiction. Therefore, $\theta \notin (I-T)C_r$. Suppose now that $f(r)=0$. Then there is a sequence $\{x_n\}\subseteq C_r$ such that $|x_n-Tx_n|\to 0$ and hence $\{x_n-Tx_n\}\to\theta$. But $\{x_n-Tx_n\}\subseteq (I-T)C_r$, a closed set. Thus we obtain $\theta \in (I-T)C_r$, contradicting our statement above that $\theta \notin (I-T)C_r$. Therefore, $f(r)>0$ for $0<r<M$.

We extend the domain of f to $[0, \infty)$ by defining $f(0)=0$ and $f(r)=\sup\{f(s):s<M\}$ for $r\geqq M$. It is easy to verify that f so defined fulfills the hypotheses of Condition I; in particular, $|x-Tx|\geqq f(d(x, F))$ for each $x \in C$. Q.E.D.

A consequence of Lemma 1 and Theorem 2 is

COROLLARY 1 (BROWDER AND PETRYSHYN [2]). *Let C be a closed, convex subset of a uniformly convex Banach space X and $T:C\to C$ a nonexpansive mapping. For $\lambda \in (0, 1)$ let T_λ be given by $T_\lambda=\lambda I+(1-\lambda)T$. (Notice that $M(x_1, 1-\lambda, T)=\{T_\lambda^n x_1\}$.) If $I-T$ maps closed bounded subsets of C onto closed subsets of X and if the set F of fixed points of T is nonempty, then for any $\lambda \in (0, 1)$ and every x in C the sequence $\{T_\lambda^n x\}$ converges to a member of F.*

A mapping $T:C\to X$ of a subset C of a Banach space X is said to be *demicompact* [11] provided whenever $\{x_n\}\subseteq C$ is bounded and $\{x_n-Tx_n\}$ converges then there is a subsequence $\{x_{n_i}\}$ which converges. If a mapping T is continuous as well as demicompact then, according to Opial [9, p. 41], the mapping $I-T$ maps closed bounded subsets of C onto closed subsets of X. In particular, if $T:C\to C$ is nonexpansive and demicompact and has a fixed point in C, it follows from Opial's result and Lemma 1 that T must satisfy Condition I. Using a different approach, Groetsch [5] has established the convergence of mean-value iterates of nonexpansive, demicompact mappings to a fixed point of the mapping.

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