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TWO UNRELATED RESULTS INVOLVING BAIRE SPACES

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ABSTRACT. Two results are obtained in this paper. The first is a generalization of J. C. Oxtoby's category analogue of the Kolmogoroff zero-one law. The second result: every dense G_δ subset of a quasi-regular α-favorable space is α-favorable.

1. **An extension of Oxtoby's category zero-one law.** Suppose X is the product of the family $(X_i)_{i\in I}$ of sets. A subset A of X is called a tail set [2] if, whenever $x=(x_i)\in A$, $y=(y_i)\in X$, and there is a finite set F such that $x_i=y_i$ whenever $i\in I\sim F$, then $y\in A$.

In [2], J. C. Oxtoby proved the following category analogue of Kolmogoroff's zero-one law in probability.

1.1 THEOREM. *Suppose each X_i is a topological space with a countable pseudo-base* [2].

1.1.1. *If A is a tail set which has the property of Baire in X, then either A or $X\sim A$ is of the first category in X.*

The following definition is convenient.

DEFINITION. A topological space $(X, \mathscr{T})$ is said to have property (P) if, whenever Y is a Baire space for which there is a nonempty U in $\mathscr{T}$ such that $U\times Y$ is a Baire space, then $V\times Y$ is a Baire space for every V in $\mathscr{T}$ for which V is a Baire space.

In this section, we shall, utilizing a slight modification of Oxtoby's proof of 1.1, prove the following statement.

1.2 THEOREM. *If each X_i has property* (P), *then* 1.1.1 *holds.*

The following statement indicates that many spaces have property (P).

1.3 PROPOSITION. *If $(X, \mathscr{T})$ satisfies any of the following conditions, then X has property* (P).

(1) *If $U\in\mathscr{T}$ and U is a Baire space, then $U\times Y$ is a Baire space whenever Y is a Baire space.*

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(1a) *$\mathcal{T}$ has a locally countable pseudo-base* [**2**].

(1b) *X is weakly α-favorable* [**4**].

(1c) *X is a Borel set in a compact, Hausdorff space T.*

(2) *If U, V are nonempty open subsets of X, then there are nonempty U', V' in $\mathcal{T}$ such that $U' \subset U$, $V' \subset V$, and U' is homeomorphic with V'.*

(2a) *X is a homogeneous space.*

(3) *X is the product of a family of spaces, each of which has property* (P).

Proof. It is clear that any space which satisfies (1) has property (P). And, it follows from Theorem 2 of [**2**] (3(9) of [**4**]), that if X satisfies (1a) ((1b)), then X satisfies (1).

Suppose (1c) holds and that $U \in \mathcal{T}$ such that U is a Baire space. Then U is a dense Borel subset of $T' = \mathrm{cl}_T\, U$. Hence U is α-favorable (see 2.4 (1)).

If (2) holds and Y is such that $U \times Y$ is a Baire space for some nonempty U in $\mathcal{T}$, then every nonempty V in $\mathcal{T}$ which is a Baire space contains a nonempty open set V' such that $V' \times Y$ is a Baire space. Hence X has property (P). And, if X satisfies (2a), clearly X satisfies (2).

The proof of (3) is easy and is omitted.

Proof of 1.2. Suppose A is of the second category in X. By the Banach category theorem, there is a nonempty basic open set U such that $U \cap A$ is a Baire space. Suppose that V is an open subset of X which is of the second category in X. By the Banach category theorem, there is a nonempty basic open set W contained in V such that W is a Baire space. There are a finite subset F of I, open sets U^*, W^* of $\prod_{i\in F} X_i$, and a subset A^* of $\prod_{i\in I\sim F} X_i$ such that $U = U^* \times \prod_{i\in I\sim F} X_i$, $W = W^* \times \prod_{i\in I\sim F} X_i$, and $A = \prod_{i\in F} X_i \times A^*$. Then $U \cap A = U^* \times A^*$ and $W \cap A = W^* \times A^*$. Since $U \cap A$ and W are Baire spaces, U^*, W^*, and A^* are Baire spaces. Since $\prod_{i\in F} X_i$ has property (P), $W \cap A$ is a Baire space. Since A is a dense subset of X with the property of Baire, this implies that $X \sim A$ is of the first category in X.

1.4 Proposition. *Suppose that, for each i in I, the set $\{j \in I : X_j$ is homeomorphic with $X_i\}$ is infinite. Then* 1.1.1 *holds.*

The proof of 1.2 establishes 1.4, if we choose U and W so that U^* is homeomorphic with W^*.

Combining 1.2, 3(4) and 3(9) of [**4**], and Theorem 3 of [**2**], we have the following generalization of Theorem 4 of [**2**].

1.5 Theorem. *Suppose each X_i is either a Baire space with a countable pseudo-base or is weakly α-favorable. Then X is a Baire space and* 1.1.1 *holds.*

REMARKS. (1) Proposition 1.4, when I is countable, follows from the theorem in [3].

(2) If, in 1.2, the X_i are arbitrary topological spaces, then 1.1.1, with the phrase "has the property of Baire in X" replaced by "is a Borel set in X", holds. For then A^* is a dense Borel subset of $\prod_{i\in I\sim F} X_i$ such that A^* is a Baire space. Hence $(\prod_{i\in I\sim F} X_i)\sim A^*$ is of the first category in $\prod_{i\in I\sim F} X_i$.

2. A topological space $(X, \mathcal{T})$ is called α-favorable [1] if there is a function $\varphi:\mathcal{T}^*\to\mathcal{T}^*$, where $\mathcal{T}^*=\{U\in\mathcal{T}: U\neq\varnothing\}$, such that $\varphi(U)\subset U$ for all U in $\mathcal{T}^*$ and, if $\gamma: N\to\mathcal{T}^*$ (here N denotes the set of natural numbers) is such that $\gamma(n+1)\subset\varphi(\gamma(n))$ for all n in N, then $\bigcap\{\gamma(n): n\in N\}\neq\varnothing$.

In [4], a related class of spaces, the class $\mathscr{C}_w$ of weakly α-favorable spaces, is considered. It is shown that $\mathscr{C}_w$ has a number of reasonable properties (3(1) through 3(10)). It would be of interest to know which of these properties the class $\mathscr{C}$ of α-favorable spaces has (it is easily verified that $\mathscr{C}$ satisfies (1), (2), (4), (5), (7), (9), and (10)). In this section we prove that $\mathscr{C}$ satisfies (6) and half of the statement obtained when the terms "α-favorable" and "weakly α-favorable" are interchanged in (3).

2.1 PROPOSITION. (a) *If $(X, \mathcal{T})$ is a dense G_δ subset of a quasi-regular, α-favorable space $(Y, \mathscr{U})$, then X is α-favorable.*

(b) *If $(X, \mathcal{T})$ is pseudo-complete* [2], *then X is α-favorable.*

PROOF. If X satisfies the hypothesis of either (a) or (b), then X satisfies the following condition.

2.2. There is a sequence $(\varphi_n)_{n\in N}$ such that, for each n in N, $\varphi_n:\mathcal{T}^*\to\mathcal{T}^*$, $\varphi_n(U)\subset U$ for all U in $\mathcal{T}^*$, and if $\gamma: N\to\mathcal{T}^*$ is such that $\gamma(n+1)\subset\varphi_n(\gamma(n))$ for all n in N, then $\bigcap\{\gamma(n): n\in N\}\neq\varnothing$.

This is easily verified if X is pseudo-complete. If X satisfies the hypothesis of (a), then a simpler version of the argument used in proving 3(6) of [4] establishes 2.2. In detail: Let $\mathscr{R}$ denote the family of all regular elements of $\mathscr{U}$. Since Y is quasi-regular, we may assume that $\varphi(U)\in\mathscr{R}$ for all U in $\mathscr{U}^*$, where φ is the function guaranteed by the definition of α-favorable. Suppose $X=\bigcap\{G_n: n\in N\}$, where $G_n\in\mathscr{U}$ for all n in N. Define $\gamma:\mathcal{T}\to\mathscr{U}$ so that $X\cap\gamma(U)=U$ for all U in $\mathcal{T}$. Define φ_n by letting $\varphi_n(U)=X\cap\varphi(\gamma(U)\cap G_n)$ for all U in $\mathcal{T}^*$. Then $(\varphi_n)_{n\in N}$ satisfies 2.2.

And, if 2.2 holds, then we may assume that $\varphi_{n+1}(U)\subset\varphi_n(U)$ for all n in N and all $U\in\mathcal{T}^*$; if necessary, replace $(\varphi_n)_{n\in N}$ by $(\psi_n)_{n\in N}$, where $\psi_1=\varphi_1$ and $\psi_{n+1}=\varphi_{n+1}\circ\psi_n$ for all n in N.

We shall show that any space which satisfies 2.2 is α-favorable. It is convenient to introduce the following condition.

2.3. There is a sequence $(\mathscr{P}_n)_{n\in N}$ of pseudo-bases for X such that, if $P_n\in\mathscr{P}_n$ for all n in N, then $\operatorname{int}[\bigcap\{P_n: n\in N\}]=\varnothing$.

Case 1. Suppose X satisfies 2.3. We may assume that (1) $X \in \mathscr{P}_1$, and (2) for each n in N, $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$. For n in N, let

$$\mathscr{B}_n = \{U \in \mathscr{T}^* : \text{There is } P_n \text{ in } \mathscr{P}_n \text{ such that } U \subset P_n\}$$

and let $\mathscr{A}_n = \mathscr{B}_n \sim \mathscr{B}_{n+1}$. Since (2) holds, $\mathscr{A}_i \cap \mathscr{A}_j = \varnothing$ if $i \neq j$. Since (1) and 2.3 hold, $\mathscr{T}^* = \bigcup \{\mathscr{A}_n : n \in N\}$. For U in $\mathscr{A}_n$, let $\varphi(U)$ be an element of $\mathscr{P}_{n+1}$ such that $\varphi(U) \subset \varphi_n(U)$. Suppose, now, that $\gamma : N \to \mathscr{T}^*$ is such that $\gamma(n+1) \subset \varphi(\gamma(n))$ for all n in N. Then there is $j : N \to N$ such that $\gamma(n) \in \mathscr{A}_{j(n)}$. And, since $\gamma(n+1) \subset \varphi(\gamma(n)) \in \mathscr{B}_{j(n)+1}$, $j(n) < j(n+1)$. Therefore $\gamma(n+1) \subset \varphi_n(\gamma(n))$ for all n in N and $\bigcap \{\gamma(n) : n \in N\} \neq \varnothing$.

Case 2. Suppose that no U in $\mathscr{T}^*$ satisfies 2.3. Suppose $U \in \mathscr{T}^*$ and define, by induction, a sequence $(\mathscr{U}_n)_{n \in N}$ such that, for each n in N, $\mathscr{U}_n \subset \{V \in \mathscr{T}^* : V \subset U\}$, $\varphi_n[\mathscr{U}_n]$ is a disjoint family such that $\bigcup \varphi_n[\mathscr{U}_n]$ is dense in U, and $\mathscr{U}_{n+1}$ refines $\varphi_n[\mathscr{U}_n]$. Since U does not satisfy 2.3, there is a sequence $(U_n)_{n \in N}$ such that, for each n in N, $U_n \in \mathscr{U}_n$, $U_{n+1} \subset \varphi_n(U_n)$, and $\operatorname{int}[\bigcap \{U_n : n \in N\}] \neq \varnothing$. Let $\varphi(U) = \operatorname{int}[\bigcap \{U_n : n \in N\}]$. Suppose $\gamma : N \to \mathscr{T}^*$ is such that $\gamma(n+1) \subset \varphi(\gamma(n))$ for all n in N. For each n, there is V_n in $\mathscr{T}^*$ such that $\varphi(\gamma(n)) \subset \varphi_n(V_n) \subset V_n \subset \gamma(n)$. So $V_{n+1} \subset \gamma(n+1) \subset \varphi(\gamma(n)) \subset \varphi_n(V_n)$, and $\bigcap \{\gamma(n) : n \in N\} = \bigcap \{V_n : n \in N\} \neq \varnothing$.

Case 3. Suppose neither Case 1 nor Case 2 holds. Then there are U, V in $\mathscr{T}^*$ such that $U \cap V = \varnothing$, $U \cup V$ is dense in X, U satisfies 2.3 and no non-empty open subset of V satisfies 2.3. Therefore U and V, and hence X, are α-favorable.

2.4 Remarks. (1) It follows from 2.1 (a) that, if X is a Baire space which is a dense subset of the quasi-regular, α-favorable space Y, and X has the property of Baire in Y, then X is α-favorable.

(2) A G_δ subset X of an α-favorable space Y need not be α-favorable even if Y is metrizable, X is a Baire space, and X is a closed subset of Y. (Let X be a subspace of the real line R which is a Baire space that is not α-favorable, and let $Y = R^2 \sim \{(x, y) \in R^2 : x \notin X, y = 0\}$.)

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