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ON THE WEAK TYPE (1, 1) INEQUALITY FOR CONJUGATE FUNCTIONS

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ABSTRACT. A theorem of Kolmogorov states that there is a positive constant K such that if $\tilde{f}$ is the conjugate function of an integrable real valued function f on the unit circle then $m\{|\tilde{f}|\geqq\lambda\}\leqq K\|f\|_1/\lambda$, $\lambda>0$. It is shown that the smallest possible value for K in this theorem, the so called weak type (1, 1) norm of the conjugate function operator, is $(1+3^{-2}+5^{-2}+\cdots)/(1-3^{-2}+5^{-2}-\cdots)\approx 1.347$. This number is also shown to be the weak type (1, 1) norm of the Hilbert transform operator on functions defined on the real line. The proof uses P. Lévy's result that Brownian motion in the plane is conformally invariant.

1. **Introduction.** Let D be the open unit disc in the complex plane C, A be the unit circle, S be the strip $\{z\in C: -1<\operatorname{Im} z<1\}$, R be the real numbers, and U be the upper half plane $\{z\in C:\operatorname{Im} z>0\}$.

Lebesgue measure on R will be denoted by μ and m will stand for linear Lebesgue measure on A divided by 2π. If f is a function on R we write $\|f\|_1=\int_R|f|\,d\mu$, and the Hilbert transform $\tilde{f}$ of integrable functions f is defined by

$$\tilde{f}(t)=\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{f(t-s)}{s}\,d\mu(s).$$

If f is a function on A we write $\|f\|_1=\int_A|f|\,dm$, and the conjugate function $\tilde{f}$ of integrable functions f is defined by

$$\tilde{f}(e^{i\theta})=PV\int_{-\pi}^{\pi}f(e^{i(\theta-\varphi)})\cot(\varphi/2)\,dm(\varphi).$$

Θ will stand for $(8/\pi^2)\int_0^\infty ae^a da/(1+e^{2a})=(8/\pi^2)(1-3^{-2}+5^{-2}-\cdots)=(1-3^{-2}+5^{-2}-\cdots)/(1+3^{-2}+5^{-2}+\cdots)$.

The following theorems will be proved.

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THEOREM 1. *If $\tilde{f}$ is the conjugate function of an integrable function f on A then*

$$m\{|\tilde{f}| \geqq \lambda\} \leqq \Theta^{-1} \|f\|_1/\lambda, \qquad \lambda > 0, \tag{1}$$

and (1) *does not hold in general if Θ^{-1} is replaced by a smaller number.*

THEOREM 2. *If $\tilde{f}$ is the Hilbert transform of an integrable function f on R then*

$$m\{|\tilde{f}| \geqq \lambda\} \leqq \Theta^{-1} \|f\|_1/\lambda, \qquad \lambda > 0, \tag{2}$$

and (2) *does not hold in general if Θ^{-1} is replaced by a smaller number.*

The existence of a positive constant K such that (1) and (2) hold with Θ^{-1} replaced by K is due to Kolmogorov. (See [**5**, p. 134].)

The smallest possible constant K for which $m\{|\tilde{f}|\geqq\lambda\}\leqq K\|f\|_1/\lambda$ is true for nonnegative functions f is known to be 1, for both conjugate functions and Hilbert transforms (see [**1**]). Pichorides [**3**] finds best constants in some strong type conjugate function and Hilbert transform inequalities.

An easy example shows that Θ^{-1} cannot be replaced by a smaller number in (1). Let $G(z)=(2/\pi)\log[(iz-1)/(z-i)]-i$. Then $G(0)=0$, G maps D onto S and maps A onto the boundary of S. Let $g(e^{i\theta})=\operatorname{Re} G(e^{i\theta})$. It can be checked that $\|g\|_1=\Theta$, and $\tilde{g}(e^{i\theta})=\operatorname{Im} G(e^{i\theta})$ so that $m\{z \in A:|\tilde{g}(z)|=1\}$. Thus, if $\lambda=1$ and $f=g$, (1) is an equality.

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2. **Preliminary lemmas.** In this section $Z_t=X_t+iY_t$, $0\leqq t<\infty$, will be a continuous two dimensional Brownian motion. If $z \in C$, P_z and E_z will stand for probability and expectation associated with Z_t given $P(Z_0=z)=1$. τ will be the first time $|Y_t|=1$. By the result of Kakutani which equates hitting probabilities of Brownian motions and harmonic measures (see [**2**]), the distribution of Z_τ under P_0 is harmonic measure on the boundary of S relative to 0. Thus the distribution of X_τ is given by the density $e^{\pi t/2}/(1+e^{\pi t})$, $-\infty<t<\infty$, so that

$$E_0\,|X_\tau| = \Theta. \tag{3}$$

LEMMA 1. *If r and s are real numbers and $-1\leqq s\leqq 1$ then*

(i) $E_{r+is}|X_\tau|\geqq E_{is}|X_\tau|$;

(ii) $E_{is}|X_\tau|\leqq\Theta$.

PROOF. (i) The distribution of X_τ under P_{is} can be found explicitly by Kakutani's theorem. It is given by a density $f_{is}(t)$, $-\infty<t<\infty$. Only the fact that f is a symmetric function around 0, which follows from the symmetry of a Brownian motion about a line through its point of origin, will be needed. The distribution of X_τ under P_{r+is} will be given by the density function $f_{r+is}(t)=f_{is}(t-r)$. Thus

$$\begin{aligned} E_{r+is}|X_\tau| &= \int_{-\infty}^{\infty} |t| f_{r+is}(t)\,dt \\ &= \int_0^\infty (|t-r| + |t+r|) f_{is}(t)\,dt \\ &\geqq 2\int_0^\infty |t| f_{is}(t)\,dt = E_{is}|X_\tau|, \end{aligned}$$

proving (i).

(ii) Let η be the first time that $|Y_t|=s$. Then under P_0, Z_η is distributed on the lines $\{z:|\mathrm{Im}\, z|=s\}$. Using (i), the strong Markov property, and the fact that the distribution of X_τ is the same under P_{is} and P_{-is}, we have

$$\begin{aligned} E_0|X_\tau| &= E_0 E_{Z_\eta}|X_\tau| \\ &\geqq E_{is}|X_\tau|/2 + E_{-is}|X_\tau|/2 \\ &= E_{is}|X_\tau|, \end{aligned}$$

proving (ii).

LEMMA 2. *Let r be a real number and μ be a stopping time for Z_t. Then* $E_r|X_{\min(\tau,\mu)}|\geqq\Theta(1-P_r(\mu<\tau))$.

PROOF. Clearly $E_{r+is}|X_\tau-r|=E_{is}|X_\tau|$. Thus, using Lemma 1 and the strong Markov property, we have

$$\begin{aligned} \Theta &= E_0|X_\tau| \leqq E_r|X_\tau| \\ &\leqq E_r|X_{\min(\mu,\tau)}| + E_r|X_\tau - X_{\min(\mu,\tau)}| \\ &\doteq E_r|X_{\min(\mu,\tau)}| + E_r E_{Z_{\min(\mu,\tau)}}|X_\tau - X_{\min(\mu,\tau)}| \\ &= E_r|X_{\min(\mu,\tau)}| + E_r E_{iY_{\min(\mu,\tau)}}|X_\tau| \\ &\leqq E_r|X_{\min(\mu,\tau)}| + \Theta P_r(\mu<\tau), \end{aligned}$$

proving (ii).

3. **Proof of Theorem 1.** In this section H_t, $0\leqq t<\infty$, will stand for a complex Brownian motion started at 0, and d will be $\inf\{t:|H_t|=1\}$. Define $F=U+iV$ on $D\cup A$ by $F(z)=\int_A[P(z,e^{i\theta})+iQ(z,e^{i\theta})]f(e^{i\theta})\,d\theta$, $z\in D$, where P and Q are the Poisson and Poisson conjugate kernels, and $F(z)=f(z)+i\tilde{f}(z)$, $z\in A$. F is analytic in D so, by a result of Levy, if $\gamma(t)=\int_0^t|F'(H_t)|^2\,dt$ then γ is almost surely strictly increasing and the process Γ_t defined for $0\leqq t\leqq\gamma(d)$ by $\Gamma_{\gamma(t)}=A_{\gamma(t)}+iB_{\gamma(t)}=F(H_t)$ is a standard Brownian motion started at $F(0)=\int_A f\,dm$ and stopped at the time $e=\gamma(d)$. [1] has a more complete discussion of this. Since H_d is uniformly distributed on A, $\|f\|_1=E|A_e|$ and $m\{|\tilde{f}|\geqq 1\}=P(|B_e|\geqq 1)$. Let $\tau=\inf\{t:|B(t)|=1\}$, and let $v_r=\inf\{t:|H_t|=r\}$. Then $|A_{\min(t,\gamma(v_r))}|$, $0\leqq t\leqq\infty$, is a continuous

nonnegative bounded submartingale so that

$$E\,|A_{\min(\tau,\gamma(v_r))}| \leqq E\,|A_{\gamma(v_r)}|$$
$$= \int_A |\mathrm{Re}\,F(re^{i\theta})|\,d\theta/2\pi \leqq \|f\|_1 = E\,|A_e|.$$

Now $\lim_{r\to 1} v_r = d$, so $\lim_{r\to 1}\gamma(v_r) = e$, and thus

$$E\,|A_{\min(\tau,e)}| \leqq \liminf_{r\to 1} E\,|A_{\min(\tau,\gamma(v_r))}| \leqq E\,|A_e|.$$

Thus, by Lemma 2,

$$m\{|\tilde f| \geqq 1\} = P(|B_e| \geqq 1) \leqq P\Big(\sup_{0\leqq t\leqq e} |B_t| = 1\Big)$$
$$= P(|B_{\min(\tau,e)}| = 1) = 1 - P(e < \tau)$$
$$\leqq \Theta^{-1} E\,|A_{\min(e,\tau)}| \leqq \Theta^{-1} E\,|A_e| = \Theta^{-1}\,\|f\|_1.$$

For general $\lambda > 0$,

$$m(|\tilde f| \geqq \lambda) = m(|(f/\lambda)^{\sim}| \geqq 1) \leqq \Theta^{-1}\,\|f/\lambda\|_1 = \Theta^{-1}\,\|f\|_1/\lambda,$$

proving (i) and completing the proof of Theorem 1.

4. **Proof of Theorem 2.** The proof of Zygmund [5, Vol. 2, p. 256] that constants in some strong type inequalities for the Hilbert transform do not exceed the constants in the related conjugate function inequalities, can be readily modified to apply to the weak type (1, 1) inequality. Thus (2) holds. This can also be proved in a manner similar to the proof of (1).

Next, examples will be given to show that Θ^{-1} cannot be replaced by a smaller number in (2). Let $H(z) = (1+z)^2/4z$. Then $H(0) = \infty$, H maps the half disc $D \cap U$ onto U and maps the boundary of $D \cap U$ onto R. Let K be the inverse of H. Then K maps $[0, 1]$ onto $\{e^{i\theta}, 0 \leqq \theta \leqq \pi\}$ and maps $R - [0, 1]$ onto $(-1, 1)$. Let d represent the density of $K([0, 1])$ on A with respect to m, i.e., if $0 < \alpha < \beta < 2\pi$, $\int_\alpha^\beta d(e^{i\theta})\,dm(e^{i\theta}) = \mu\{r \in [0, 1]: K(r) \in \{e^{i\theta}, \alpha < \theta < \beta\}\}$. We will not need the exact form of d but only that it is a continuous function on A, which is easily checked. Let d_n be the density of $K^n([0, 1])$ on A, where the n indicates product.

LEMMA 3. *As n approaches infinity, d_n approaches* 1 *uniformly on A.*

PROOF. Let $0 \leqq \theta \leqq \varphi < 2\pi$.
We have

$$|d_n(e^{i\theta}) - d_n(e^{i\varphi})| = \left| n^{-1}\sum_{k=0}^{n-1} d\left(\exp\left[\frac{2k\pi + \theta}{n}\right]\right) - n^{-1}\sum_{k=0}^{n-1} d\left(\exp\left[\frac{2k\pi + \varphi}{n}\right]\right)\right|$$
$$\leqq \sup_{0\leqq k<n} \left| d\left(\exp\left[\frac{2k\pi + \theta}{n}\right]\right) - d\left(\exp\left[\frac{2k\pi + \varphi}{n}\right]\right)\right|,$$

which approaches 0 as n approaches infinity by the uniform continuity of d. Since $\int_A d_n\, dm=\mu[0, 1]=1$, this establishes Lemma 3.

Now let G be as in the introduction. G maps D onto S, maps A onto the boundary of S, maps $[-1, 1]$ onto $[-i, i]$, and $G(0)=0$. Thus if we define M_n by $M_n(z)=G(K^n(z))$, M_n maps U onto S, maps $[0, 1]$ onto the boundary of S, and maps $R-[0, 1]$ onto $(-i, i)$. Define $m_n(r)=\operatorname{Re} M_n(r)$, $-\infty<r<\infty$. Then since $M_n(r)=0$ if $r\notin[0, 1]$, Lemma 3 implies

$$\|m_n\|_1=\int_{[0,1]}|m_n(t)|\, d\mu(t)=\int_A|\operatorname{Re} G(e^{i\theta})|\, d_n(e^{i\theta})\, dm(e^{i\theta})$$
$$\rightarrow\int_A|\operatorname{Re} G(e^{i\theta})|\, dm(e^{i\theta})=\Theta.$$

Since $M_n(z)\rightarrow 0$ as $|z|\rightarrow\infty$ it follows that $\tilde{m}_n(r)=\operatorname{Im} M_n(r)$, $-\infty<r<\infty$, and thus $\mu\{|\tilde{m}_n|\geqq 1\}=\mu\{[0, 1]\}=1$. Thus Θ^{-1} cannot be replaced by a smaller number in (2).

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