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SOME PROPERTIES OF SELF-INVERSIVE POLYNOMIALS

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ABSTRACT. A complex polynomial is called self-inversive [5, p. 201] if its set of zeros (listing multiple zeros as many times as their multiplicity indicates) is symmetric with respect to the unit circle. We prove that if P is self-inversive and of degree n then $\|P'\|=\frac{1}{2}n\|P\|$ where $\|P'\|$ and $\|P\|$ denote the maximum modulus of P' and P, respectively, on the unit circle. This extends a theorem of P. Lax [4]. We also show that if $P(z)=\sum_{j=0}^{n} a_j z^j$ has all its zeros on $|z|=1$ then $2\sum_{j=0}^{n}|a_j|^2\leqq\|P\|^2$. Finally, as a consequence of this inequality, we show that when P has all its zeros on $|z|=1$ then $2^{1/2}|a_{n/2}|\leqq\|P\|$ and $2|a_j|\leqq\|P\|$ for $j\neq n/2$. This answers in part a question presented in [3, p. 24].

1. **Main theorems.** We begin with a

DEFINITION. A polynomial P with zeros $z_1, z_2, \cdots, z_n$ is self-inversive if $\{z_1, z_2, \cdots, z_n\}=\{1/\bar{z}_1, 1/\bar{z}_2, \cdots, 1/\bar{z}_n\}$.

Some properties of self-inversive polynomials are given by the following lemmas. These properties have been noted by other authors (see for example [1] and [5, p. 204]). In what follows, if $P(z)=\sum_{j=0}^{n} a_j z^j$ then $\bar{P}(z)$ denotes $\sum_{j=0}^{n} \bar{a}_j z^j$.

LEMMA 1. *If $P(z)=\sum_{j=0}^{n} a_j z^j$, $a_n\neq 0$, then the following are equivalent:*

(i) P is self-inversive.

(ii) $\bar{a}_n P(z)=a_0 z^n \bar{P}(1/z)$ for each complex number z.

(iii) $a_0\bar{a}_j=\bar{a}_n a_{n-j}$; $j=0, 1, \cdots, n$.

This lemma follows easily from the previous definition.

LEMMA 2. *If P is self-inversive and $P(z)=\sum_{j=0}^{n} a_j z^j$, $a_n\neq 0$, then*

$$\text{(i)} \qquad \bar{a}_n[nP(z) - zP'(z)] = a_0 z^{n-1}\bar{P}'(1/z) \quad \textit{for each } z,$$

and

$$\text{(ii)} \qquad |nP(z)/zP'(z) - 1| = 1 \quad \textit{for each } z \textit{ on } |z| = 1.$$

PROOF. By the previous lemma we can write: $\bar{a}_n P(z)=a_0 z^n \sum_{j=0}^{n} \bar{a}_j z^{-j}$. We obtain (i) by differentiating this last identity. Then (ii) follows from

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(i) by noting that $1/\bar{z}=z$ when $|z|=1$, and $|a_0|=|a_n|$. We remark that Lemma 2 implies that if P is self-inversive then P' has no zeros on $|z|=1$ except at the multiple zeros of P, a result that is proved by other means in [**5**, p. 205].

By a circular region is meant the image of the unit disk (open or closed) under a bilinear transformation. We shall need the following theorem of DeBruijn [**2**].

THEOREM. *Let K be a circular region and let P be any polynomial of degree n. If $u \in K$ and $Q(z)=n^{-1}[nP(z)+(u-z)P'(z)]$ then $Q(K)\subseteq P(K)$.*

Throughout the rest of this paper we shall use the notation $\|P\|$, P a complex polynomial, to denote the maximum modulus of P on the unit circle. The next theorem extends the result of P. Lax given in [**4**].

THEOREM 1. *If P is a self-inversive polynomial of degree n then $\|P'\|=\frac{1}{2}n\|P\|$.*

PROOF. Let ε be a point on $|z|=1$ such that $\|P'\|=|P'(\varepsilon)|$. Choose u on $|z|=1$ so that $nP(\varepsilon)-\varepsilon P'(\varepsilon)$ and $uP'(\varepsilon)$ have the same argument. Then by DeBruijn's theorem we have

$$|nP(\varepsilon) - \varepsilon P'(\varepsilon) + uP'(\varepsilon)| \leqq n\,\|P\|,$$

and hence

$$|nP(\varepsilon) - \varepsilon P'(\varepsilon)| + |P'(\varepsilon)| \leqq n\,\|P\|.$$

By Lemma 2, $|nP(\varepsilon)-\varepsilon P'(\varepsilon)|=|P'(\varepsilon)|$, and so $2\|P'\|\leqq n\|P\|$. To reverse this inequality we again use Lemma 2 to obtain that if $|z|=1$ then $n|P(z)|\leqq 2|P'(z)|$ and hence $n\|P\|\leqq 2\|P'\|$.

Next we consider the following conjecture presented in [**3**, p. 24]: *If P has all its zeros on $|z|=1$ and $P(z)=\sum_{j=0}^{n} a_j z^j$ then $2|a_j|\leqq\|P\|$ for* $j=0, 1, \cdots, n$. We prove the conjecture when the degree n is odd; when n is even we show that $2|a_j|\leqq\|P\|$ for $j\neq n/2$, and $2^{1/2}|a_{n/2}|\leqq\|P\|$. In the final section we show that the estimate, $2|a_{n/2}|\leqq\|P\|$ for n even, is equivalent to the above conjecture. The validity of this estimate is not established by this paper; however, some partial results are presented.

We need

THEOREM 2. *If P is self-inversive and of degree n, and $\sum_{j=-\infty}^{+\infty} c_j z^j$ is the Laurent expansion about 0 of $nP(z)/zP'(z)$ in some annulus that contains the unit circle (see the remark following Lemma 2), then*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{nP(e^{i\theta})}{P'(e^{i\theta})}\right|^2 d\theta = 2\,\mathrm{Re}(c_0).$$

In particular if all the zeros of P lie on $|z|=1$ *then* $c_0=1$ *and*

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{nP(e^{i\theta})}{P'(e^{i\theta})}\right|^2 d\theta = 2.$$

PROOF. Multiplying (i) of Lemma 2 by $\bar{P}(\bar{z})$ and then using (ii) of Lemma 1, we obtain that

$$|nP(z)/P'(z)|^2 = 2\ \mathrm{Re}[nP(z)/zP'(z)] \quad \text{for } |z| = 1.$$

The first part of the theorem now follows since

$$c_0 = \frac{1}{2\pi}\int_0^{2\pi}\frac{nP(e^{i\theta})}{e^{i\theta}P'(e^{i\theta})}\,d\theta.$$

When P has all its zeros on the unit circle then the Gauss-Lucas theorem implies that P' has all its zeros in $|z|\leqq 1$. Therefore the function defined by $nP(z)/[zP'(z)]$ is analytic in $|z|\geqq 1$ and at $z=\infty$, and hence $c_0=\lim_{z\to\infty} nP(z)/[zP'(z)]=1$.

COROLLARY 1. *If P is self-inversive and* $P(z)=\sum_{j=0}^n a_j z^j$, $a_n\neq 0$, *then* $2\sum_{j=0}^n |a_j|^2\leqq \|P\|^2\mathrm{Re}(c_0)$. *In particular if P has all its zeros on* $|z|=1$ *then* $2\sum_{j=0}^n |a_j|^2\leqq \|P\|^2$. *Moreover these inequalities are equalities if and only if the zeros of P are rotations of the nth roots of unity.*

PROOF. By applying Parseval's identity and Theorem 1, we obtain:

$$\begin{aligned}\sum_{j=0}^n |a_j|^2 &= \frac{1}{2\pi}\int_0^{2\pi}|P'(e^{i\theta})|^2\left|\frac{P(e^{i\theta})}{P'(e^{i\theta})}\right|^2 d\theta\\ &\leqq \left[\frac{\|P'\|}{n}\right]^2\left[\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{nP(e^{i\theta})}{P'(e^{i\theta})}\right|^2 d\theta\right]\\ &= \tfrac{1}{2}\,\|P\|^2\ \mathrm{Re}(c_0).\end{aligned}$$

Clearly the inequality above is equality if and only if $|P'(z)|$ is constant for $|z|=1$ or, in other words, if and only if $P(z)=a_0+a_nz^n$ where (since P is self-inversive) $|a_0|=|a_n|$.

Using Corollary 1 we now prove the following theorem which answers in part the previously mentioned conjecture.

THEOREM 3. *If P has all its zeros on* $|z|=1$ *and* $P(z)=\sum_{j=0}^n a_j z^j$, $a_n\neq 0$, *then* $2|a_j|\leqq \|P\|$ *for each* $j\neq n/2$ *and* $2^{1/2}|a_{n/2}|\leqq \|P\|$.

PROOF. From (iii) of Lemma 1 we get that $|a_j|=|a_{n-j}|, j=0, 1, \cdots, n$. Therefore if $j\neq n/2$ then $4|a_j|^2=2[|a_j|^2+|a_{n-j}|^2]\leqq 2\sum_{j=0}^n |a_j|^2\leqq \|P\|^2$. For n even, the estimate, $2^{1/2}|a_{n/2}|\leqq \|P\|$, also follows immediately from Corollary 1.

2. **Remarks concerning the middle coefficient.** Throughout this section P will denote an arbitrary self-inversive polynomial (unless further restrictions are noted) with $P(z)=\sum_{j=0}^{n} a_j z^j$, $a_n \neq 0$. Also if P has all its zeros on $|z|=1$ and n is even, then we shall refer to the estimate, $2|a_{n/2}| \leqq \|P\|$, as the middle coefficient conjecture. We present here a few remarks which are pertinent to this conjecture.

A. By using Lemma 1 it is not hard to show that the modulus of the middle coefficient of P^2 is equal to $\sum_{j=0}^{n} |a_j|^2$. Therefore, had we been able to establish the truth of the middle coefficient conjecture, then the second inequality of Corollary 1 (and hence Theorem 3) would have followed immediately. In other words, *the middle coefficient conjecture is equivalent to the conjecture mentioned in the previous section.*

B. Suppose n is even and choose λ, $|\lambda|=1$, so that $\lambda^2=\bar{a}_n/a_0$. Then by applying (iii) of Lemma 1 we obtain that for $|z|=1$, $\lambda P(z)z^{-n/2}=\operatorname{Re}[Q(z)]$, where Q is a polynomial of degree $n/2$ with leading coefficient, $2\lambda a_n$, and constant coefficient, $\lambda a_{n/2}$. Therefore if $|a_{n/2}| \leqq 2|a_n|$ then Q has a zero in $|z| \leqq 1$. It then follows that $\operatorname{Re}[Q(z)]$ vanishes somewhere on $|z|=1$. Thus we have established the following: *if n is even and if $|a_{n/2}| \leqq 2|a_n|$ then P has a zero (and hence at least two) on $|z|=1$.*

C. If n is even the polynomials defined by $\lambda P(z) \pm (\|P\|+\varepsilon)z^{n/2}$, $\varepsilon > 0$, are self-inversive by (iii) of Lemma 1. Clearly they do not vanish on $|z|=1$ and so by the previous remark $|\lambda a_{n/2} \pm (\|P\|+\varepsilon)| > 2|a_n|$. Since $|a_{n/2}| \leqq \|P\|$, ε was arbitrary, and $\lambda a_{n/2}$ is real, it follows that $|a_{n/2}|+2|a_n| \leqq \|P\|$. Therefore we have proved: *if n is even then $|a_{n/2}|+2|a_n| \leqq \|P\|$; in particular the middle coefficient conjecture is true in those cases where $|a_{n/2}| \leqq 2|a_n|$.*

D. Suppose n is even and $P(z)=a_n \prod_{j=1}^{n} (z-e^{i\theta_j})$. We shall omit the details, but by using the identity $e^{i\theta}-e^{it}=2i \sin[(\theta-t)/2]e^{i(\theta+t)/2}$ we can establish the following: *the middle coefficient conjecture is equivalent to the estimate*

$$\left| \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, d\theta \right| \leqq \max_{\theta} |f(\theta)|,$$

where f is the trignometric polynomial of degree $n/2$ defined by $f(\theta)=\prod_{j=1}^{n} \sin((\theta-\theta_j)/2)$.

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