

## ON THE A.E. CONVERGENCE OF WALSH-KACZMARZ-FOURIER SERIES

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**ABSTRACT.** It is shown that partial sums of Walsh-Kaczmarz-Fourier series of functions in the Orlicz class  $L(\log^+ L)^2$  converge a.e. The proof utilizes an estimate of P. Sjölin on the partial sums of the usual Walsh-Fourier series.

The Walsh-Kaczmarz system is a reordering of the usual Walsh system within dyadic blocks of indices  $2^N$  to  $2^{N+1}$ ,  $N=0, 1, \dots$ . The a.e. convergence properties of Fourier series with respect to this system have been investigated by L. A. Balashov [1] and K. H. Moon [7]. Balashov showed that there exist functions in the Orlicz class  $L(\log^+ L)^{1-\epsilon}$ ,  $0 < \epsilon < 1$ , whose Walsh-Kaczmarz-Fourier series diverge a.e. Moon established the a.e. convergence of Walsh-Kaczmarz-Fourier series of  $L^2$  functions. In this note we prove, using a theorem of P. Sjölin [9] on the a.e. convergence of Walsh-Fourier series, the a.e. convergence result for functions in the class  $L(\log^+ L)^2$ .

The author would like to thank Professor Richard A. Hunt for many helpful conversations.

We recall the definition of the Walsh system  $\{w_n\}$ . Let  $r_n$ , where  $r_n(x) = \text{sgn}(\sin 2^{n+1}\pi x)$ , be the  $n$ th Rademacher function. For any non-negative integer  $n$ , with dyadic expansion  $n = \sum_{j=0}^{\infty} \epsilon_j 2^j$ ,  $w_n = \prod_{j=0}^{\infty} r_j^{\epsilon_j}$ .

The Walsh-Kaczmarz system  $\{\phi_n\}$  is defined as follows:  $\phi_0 = 1$ ,  $\phi_1 = r_0$ , and for  $N=1, 2, \dots$ ,  $2^N \leq n < 2^{N+1}$ , with  $n = \sum_{j=0}^N \epsilon_j 2^j$ , where  $\epsilon_j = 0$  or  $1$  if  $0 \leq j \leq N-1$ , and  $\epsilon_N = 1$ ,  $\phi_n = r_N \prod_{j=0}^{N-1} r_{N-j-1}^{\epsilon_j}$ . The system  $\{\phi_n\}$  so defined is a rearrangement of  $\{w_n\}$  within dyadic blocks of indices  $2^N \leq n < 2^{N+1}$ ,  $N=1, 2, \dots$ .

For  $f \in L^1(0, 1)$ , let  $S_n f = \sum_{j=0}^{n-1} \phi_j \int_0^1 f \phi_j dt$  denote the  $n$ th partial sum of the Fourier series of  $f$  with respect to the Walsh-Kaczmarz system  $\{\phi_n\}$ .

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Received by the editors July 27, 1973.

AMS (MOS) subject classifications (1970). Primary 42A56; Secondary 42A20, 46E30.

Key words and phrases. Walsh-Kaczmarz-Fourier series, Walsh-Fourier series, a.e. convergence, Orlicz space.

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**THEOREM.** *If  $\int_0^1 |f|(\log^+ |f|)^2 dx < \infty$ , then  $S_n f$  converges to  $f$  a.e.*

We will show that there exist absolute constants  $C_1$  and  $C_2$  such that

$$(1) \quad m\left\{\sup_n |S_n f| > y\right\} \leq y^{-1} \left( C_1 \int_0^1 |f|(\log^+ |f|)^2 dx + C_2 \right)$$

for all  $y > 0, f \in L(\log^+ L)^2$ . The Theorem will follow from (1) by the usual density argument.

Before we proceed to prove (1) we need to make the following observation. Let  $\tau$  be a permutation of the set of all nonnegative integers. An ordering  $\{\theta_n\}$  of the Walsh functions is said to be the Paley ordering generated by  $\{r_{\tau(n)}\}$  if for any nonnegative integer  $n$  with dyadic expansion  $n = \sum_{j=0}^{\infty} \varepsilon_j 2^j, \theta_n = \prod_{j=0}^{\infty} r_{\tau}^{\varepsilon_j}$ . We will need several properties of the partial sums  $R_n f$  of the Fourier series of  $f$  with respect to  $\{\theta_n\}$ . These facts can easily be deduced from the corresponding ones of the Walsh-Fourier series since there is a 1-1 measure-preserving transformation  $E$  from  $(0, 1)$  onto itself such that  $r_{\tau(N)}(x) = r_N(Ex)$  a.e.,  $N = 0, 1, \dots$ , and hence  $\theta_n(x) = w_n(Ex)$  a.e.,  $n = 0, 1, \dots$ .

First, if  $n = \sum_{j=0}^{\infty} \varepsilon_j 2^j$ , it follows from the definition that

$$R_n f = \theta_n \sum_{j=0}^{\infty} \varepsilon_j (R_{2^{j+1}}(\theta_n f) - R_{2^j}(\theta_n f)).$$

(See Paley [8].) Now, for any  $g \in L^1, R_{2^j}(g)$  is the average of  $g$  over sets of the form  $\{r_{\tau(0)} = c_0, \dots, r_{\tau(j-1)} = c_{j-1}\}$  where  $c_k = \pm 1, k = 0, \dots, j-1$ , or, in terms of conditional expectation,

$$(2) \quad R_{2^j}(g) = E(g \mid r_{\tau(0)}, \dots, r_{\tau(j-1)}), \quad j = 1, 2, \dots$$

Thus we have

$$(3) \quad R_n f = \theta_n \sum_{j=0}^{\infty} \varepsilon_j (E(\theta_n f \mid r_{\tau(0)}, \dots, r_{\tau(j)}) - E(\theta_n f \mid r_{\tau(0)}, \dots, r_{\tau(j-1)})).$$

In the above equation, when  $j=0$  the term  $E(\theta_n f \mid r_{\tau(0)}, \dots, r_{\tau(j-1)})$  is interpreted as the integral  $\int_0^1 \theta_n f dt$ .

Similarly, if  $2^N \leq n < 2^{N+1}, n = \sum_{j=0}^N \varepsilon_j 2^j$ ,

$$R_n f - R_{2^N} f = r_{\tau(N)} R_{n-2^N}(r_{\tau(N)} f)$$

$$(4) \quad = \theta_n \sum_{j=0}^{N-1} \varepsilon_j (E(\theta_n f \mid r_{\tau(0)}, \dots, r_{\tau(j)}) - E(\theta_n f \mid r_{\tau(0)}, \dots, r_{\tau(j-1)})).$$

Finally, from a theorem of Sjölín [9], we have

$$(5) \quad \int_0^1 \sup_n |R_n f| dx \leq C_1 \int_0^1 |f|(\log^+ |f|)^2 dx + C_2,$$

where  $C_1$  and  $C_2$  are absolute constants.

We now return to the proof of (1). We note that

$$\sup_n |S_n f| \leq \sup_N \sup_{2^N \leq n < 2^{N+1}} |S_n f - S_{2^N} f| + \sup_N |S_{2^N} f|.$$

Since  $S_{2^N} f$  coincides with the  $2^N$ th partial sum of the Walsh-Fourier series, (2) and Doob's inequality [10, p. 91] give

$$(6) \quad m\left\{\sup_N |S_{2^N} f| > y\right\} = m\left\{\sup_N |E(f | r_0, \dots, r_{N-1})| > y\right\} \leq y^{-1} \int_0^1 |f| dx.$$

Hence it is sufficient to prove that for every positive integer  $N_0$ ,

$$(7) \quad m\left\{\sup_{N \leq N_0} \sup_{2^N \leq n < 2^{N+1}} |S_n f - S_{2^N} f| > y\right\} \leq y^{-1} \left( C_1 \int_0^1 |f| (\log^+ |f|)^2 dx + C_2 \right).$$

To this end we observe that for  $2^N \leq n < 2^{N+1}$ ,  $N=1, \dots, N_0$ ,  $\phi_n$  is equal to the  $n$ th term of the Paley ordering of the Walsh functions generated by the sequence  $r_{N-1}, r_{N-2}, \dots, r_0, r_N, r_{N+1}, \dots$ . Hence, it follows from (4) that for  $2^N \leq n < 2^{N+1}$ , with  $n = \sum_{j=0}^N \epsilon_j(n) 2^j$ ,

$$(8) \quad S_n f - S_{2^N} f = \phi_n \sum_{j=0}^{N-1} \epsilon_j(n) (E(\phi_n f | r_{N-1}, \dots, r_{N-j-1}) - E(\phi_n f | r_{N-1}, \dots, r_{N-j})).$$

Again, in the above equation, when  $j=0$  the term  $E(\phi_n f | r_{N-1}, \dots, r_{N-j})$  is interpreted as the integral  $\int_0^1 \phi_n f dt$ .

At this point we observe that for any  $L^1$  function  $g$ , and any integers  $n, m, l \geq 0$ ,

$$(9) \quad E(g | r_n, \dots, r_{n+m}) = E(E(g | r_n, \dots, r_{n+m+l}) | r_0, \dots, r_{n+m}).$$

To see this we first note that

$$E(g | r_n, \dots, r_{n+m}) = E(E(g | r_n, \dots, r_{n+m+l}) | r_n, \dots, r_{n+m}).$$

The equality

$$E(E(g | r_n, \dots, r_{n+m+l}) | r_n, \dots, r_{n+m}) = E(E(g | r_n, \dots, r_{n+m+l}) | r_0, \dots, r_{n+m})$$

is a consequence of the independence of the Rademacher functions  $\{r_n\}$  and the following fact: (See, for example, [3, p. 285].)

Suppose  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$  are three Borel fields such that  $\mathcal{F}_1 \vee \mathcal{F}_2$ , the Borel field generated by  $\mathcal{F}_1 \cup \mathcal{F}_2$ , is independent of  $\mathcal{F}_3$ . Then, for each integrable,  $\mathcal{F}_1$ -measurable function  $h$ , we have  $E(h | \mathcal{F}_2) = E(h | \mathcal{F}_2 \vee \mathcal{F}_3)$ .

Moreover, by the independence of the Rademacher functions, we have

$$(10) \quad \int_0^1 f\phi_n dt = E(E(f\phi_n | r_N, \dots, r_{N_0}) | r_0, \dots, r_{N-1}).$$

Substituting (9) and (10) into (8), we obtain

$$S_n f - S_{2^N} f = \phi_n E \left( \sum_{j=0}^{N-1} \varepsilon_j(n) (E(f\phi_n | r_{N-j-1}, r_{N-j}, \dots, r_{N_0}) - E(f\phi_n | r_{N-j}, r_{N-j+1}, \dots, r_{N_0})) \mid r_0, \dots, r_{N-1} \right)$$

Now we consider the Paley ordering  $\{\psi_n\}$  of the Walsh functions generated by the sequence  $r_{N_0}, r_{N_0-1}, \dots, r_0, r_{N_0+1}, r_{N_0+2}, \dots$ . For each  $2^N \leq n < 2^{N+1}$ , there corresponds an integer  $m = m(n)$  such that

$$\phi_n = r_N \sum_{j=0}^{N-1} r_{N-j-1}^{\varepsilon_j(n)} = \psi_m.$$

In fact, we have  $m = \sum_{j=0}^{N_0} \eta_j(m) 2^j$ , where

$$\eta_j(m) = \begin{cases} 0 & \text{if } j < N_0 - N, \\ 1 & \text{if } j = N_0 - N, \\ \varepsilon_{j-N_0+N-1}(n) & \text{if } j > N_0 - N. \end{cases}$$

Therefore,

$$\begin{aligned} S_n f - S_{2^N} f &= \psi_m E \left( \sum_{j=N_0-N+1}^{N_0} \eta_j(m) (E(f\psi_m | r_{N_0}, \dots, r_{N_0-j}) - E(f\psi_m | r_{N_0}, \dots, r_{N_0-j+1})) \mid r_0, \dots, r_{N-1} \right) \\ &= \psi_m E \left( \sum_{j=0}^{N_0} \eta_j(m) (E(f\psi_m | r_{N_0}, \dots, r_{N_0-j}) - E(f\psi_m | r_{N_0}, \dots, r_{N_0-j+1})) \mid r_0, \dots, r_{N-1} \right) \\ &\quad - \psi_m E((E(f\psi_m | r_{N_0}, \dots, r_N) - E(f\psi_m | r_{N_0}, \dots, r_{N+1})) \mid r_0, \dots, r_{N-1}). \end{aligned}$$

The last term vanishes since the independence of the Rademacher functions implies

$$\begin{aligned} E(E(f\psi_m | r_{N_0}, \dots, r_N) | r_0, \dots, r_{N-1}) &= \int_0^1 f\psi_m dt \\ &= E(E(f\psi_m | r_{N_0}, \dots, r_{N+1}) | r_0, \dots, r_{N-1}). \end{aligned}$$

Also, from (3), if  $T_n f$  is the  $n$ th partial sum of the Fourier series of  $f$  with respect to  $\{\psi_n\}$ ,

$$\sum_{j=0}^{N_0} \eta_j(m) (E(f\psi_m | r_{N_0}, \dots, r_{N_0-j}) - E(f\psi_m | r_{N_0}, \dots, r_{N_0-j+1})) = \psi_m(T_m f).$$

Hence, for  $2^N \leq n < 2^{N+1}$ ,  $N \leq N_0$ ,

$$S_n f - S_{2^N} f = \psi_m E(\psi_m T_m f | r_0, \dots, r_{N-1}).$$

Consequently,

$$\begin{aligned} \sup_{2^N \leq n < 2^{N+1}} |S_n f - S_{2^N} f| &\leq \sup_{2^N \leq n < 2^{N+1}} E(|T_m(n) f| | r_0, \dots, r_{N-1}) \\ &\leq E\left(\sup_k |T_k f| | r_0, \dots, r_{N-1}\right) \end{aligned}$$

for all  $N \leq N_0$ . Therefore

$$\begin{aligned} m \left\{ \sup_{N \leq N_0} \sup_{2^N \leq n < 2^{N+1}} |S_n f - S_{2^N} f| > y \right\} \\ &\leq m \left\{ \sup_{N \leq N_0} E\left(\sup_k |T_k f| | r_0, \dots, r_{N-1}\right) > y \right\} \\ &\leq y^{-1} \int_0^1 \sup_k |T_k f| dx \\ &\leq y^{-1} \left( C_1 \int_0^1 |f| (\log^+ |f|)^2 dx + C_2 \right). \end{aligned}$$

Here we have made use of Doob's inequality (see (6)) and (5). This proves (7) and thus completes the proof of the Theorem.

REMARKS. For the usual Walsh-Fourier series, there is a gap between a.e. convergence results and a.e. divergence results. It is known that the Walsh-Fourier series converge a.e. for functions in the Orlicz class  $L(\log^+ L) \log^+ \log^+ L$  (Sjölin [9]), and that there are functions in the class  $L(\log^+ \log^+ L)^{1-\varepsilon}$  ( $0 < \varepsilon < 1$ ) whose Walsh-Fourier series diverge a.e. (Moon [7]). Such a gap also exists in the Walsh-Kaczmarz-Fourier series, where we have a.e. convergence for the class  $L(\log^+ L)^2$  and a.e. divergence for the class  $L(\log^+ L)^{1-\varepsilon}$  ( $0 < \varepsilon < 1$ ) (Balashov [1]).

Another proof involving modifications of the Carleson-Hunt technique [2], [5], [6] and estimates on maximal functions of the Hardy-Littlewood type yields a.e. convergence results for functions in the smaller class  $L(\log^+ L)^2 \log^+ \log^+ L$ . That proof, however, also works for more general rearrangements. It also gives various estimates on  $\sup_n |S_n f|$ . (See [11], [12] and [4].)

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