ON CYCLIC VECTORS OF WEIGHTED SHIFTS

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ABSTRACT. Sufficient conditions on a sequence are given in order that the linear manifold spanned by its right translates is dense in certain Hilbert spaces of sequences.

1. **Introduction.** A vector x in a Hilbert space H is said to be a cyclic vector for a bounded operator T on H if the sequence x, Tx, T^2x, $\cdots$ spans H. The search for cyclic vectors for certain operators leads to interesting approximation problems in function theory. One such operator is the forward shift in the space of one-sided sequences square-summable with respect to a weight. We consider a weight sequence $\{\rho(n)\}$, $n=0$, $1, \cdots$, satisfying the conditions

$$(1) \qquad\qquad \rho(n) > 0$$

$$(2) \qquad\qquad \sup_{n} \frac{\rho(n+1)}{\rho(n)} < \infty.$$

Let $l^2(\rho)$ be the Hilbert space of all one-sided sequences $\{a_n\}$ for which the norm

$$\|a\| = \left[\sum_{n=0}^{\infty} |a_n|^2 \, \rho(n) \right]^{1/2}$$

is finite. The shift operator S on $l^2(\rho)$ is defined by

$$S(a_0, a_1, \cdots) = (0, a_0, a_1, \cdots).$$

If (2) is satisfied, the operator S is bounded and

$$\|S\| = \sup[\rho(n+1)/\rho(n)]^{1/2}.$$

(See [5, Chapter 10].)

The problem of deciding whether a vector a in $l^2(\rho)$ is a cyclic vector for S is the problem of deciding whether the right translates of a span the

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space $l^2(\rho)$. Motivated by classical results in harmonic analysis we intro-
duce the "Laplace transform" of the sequence a and try to link the cyclicity
of a to properties of its transform. (See the introduction of [3].) Let us
assume that the number

$$(3) \qquad\qquad R = \liminf \rho(n)^{1/2n}$$

is different from zero. For each a in $l^2(\rho)$ define the function f_a by the
relation

$$(4) \qquad\qquad f_a(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Then f_a is analytic in the disk $\{z : |z| < R\}$. The nonvanishing of f_a is
easily seen to be a necessary condition for the cyclicity of a.

In [2] Beurling gave a complete description of cyclic vectors for the
forward shift in the case in which $\rho(n) = 1$ for every n. It was proved there
that a vector a is cyclic if and only if the associated analytic function is
outer. We must also mention the papers of Shapiro [9], [10], [11] where the
problem is studied in weighted sequence spaces. The weight sequences
that Shapiro considers are essentially moment sequences of measures on
the unit interval. In that case the problem of the density of linear combi-
nations of translates of a sequence becomes a problem of polynomial
approximation in a weighted L^2 space. The main feature of the present
work is that we do not assume that the weight sequence is a moment
sequence.

2. **The main result.** Before we state the main result of this paper, we
have to introduce some more notation. H^∞ will denote the algebra of all
bounded analytic functions in the unit disk. Let N be the Nevanlinna class
of analytic functions of bounded characteristic in the unit disk. N^+
denotes the subclass of N consisting of those functions f for which

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta \to \int_0^{2\pi} \log^+ |f(e^{i\theta})|\, d\theta$$

as $r \to 1-$, where $f(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta})$. It is well known that a function f
analytic in the unit disk belongs to the class N^+ if and only if it is expres-
sible in the form $f = \varphi/\psi$ where φ and ψ are in H^∞ and ψ is outer. (See
[4, p. 26].)

THEOREM 1. *Let $\{\rho(n)\}$, $n = 0, 1, 2, \cdots$, be a nonincreasing sequence
of positive numbers satisfying the condition*

$$(5) \qquad\qquad \liminf \rho(n)^{1/2n} = 1.$$

Let $a=(a_0, a_1, \cdots)$ be an element of $l^2(\rho)$ and let f_a be the analytic function defined by (4). *If $1/f_a$ is in the class N^+, then a is a cyclic vector for the forward shift on $l^2(\rho)$.*

REMARK. Condition (5) of the theorem deserves some comment. By elementary arguments, it can be shown that

$$\lim \inf \rho(n)^{1/n} \leqq \sup(\rho(\dot{n} + 1)/\rho(n)).$$

The theorem can actually be proved under the assumptions:

(i) $\{\rho(n)\}$ is nonincreasing.

(ii) $\lim \inf \rho(n)^{1/n}=\sup(\rho(n+1)/\rho(n))$.

(iii) The function $1/f_a$ is in the class N^+ in the disk $\{z:|z|<R\}$, where R is defined by (3).

In the version that is stated above we have thus made the normalization that both the numbers involved in (ii) are equal to one. It should also be noted that if (ii) holds, then the lim inf in (ii) can be replaced by the limit.

We need the following lemma in the proof of the theorem. This lemma is of some independent interest. It must be compared with Lemma 5 of Shields and Wallen [12]. We must mention that in the proof of the lemma we do not assume that $\lim \inf \rho(n)^{1/n}>0$.

LEMMA. *Let $\{\rho(n)\}$, $n=0, 1, 2, \cdots$, be a nonincreasing sequence of positive numbers. Let $\alpha=(\alpha_0, \alpha_1, \cdots)$ be the sequence of Fourier coefficients of an element φ of H^∞. Then the mapping M_φ defined on $l^2(\rho)$ by $M_\varphi a= \alpha * a$ is a bounded linear transformation. (Here $\alpha * a$ denotes the convolution of α with a.) The mapping $\varphi \to M_\varphi$ has the following properties.*

(i) *$\|M_\varphi\| \leqq \|\varphi\|_\infty$, where $\|\varphi\|_\infty$ is the supremum norm of φ in H^∞.*

(ii) *If $\{\varphi_n\}$ is a sequence in H^∞ such that $\{\|\varphi_n\|_\infty\}$ is bounded and $\varphi_n(e^{i\theta}) \to \varphi(e^{i\theta})$ for a.e. θ in $[0, 2\pi)$ and φ is in H^∞, then $\{M_{\varphi_n}\}$ converges to M_φ in the strong operator topology.*

PROOF. This lemma will be deduced from Theorem 1.1 of [7] using a duality argument. Let $\tau(n)=1/\rho(n)$. Then $\{\tau(n)\}$ is a nondecreasing sequence and the dual space of $l^2(\rho)$ can be identified with the space $l^2(\tau)$. The duality is given by

$$\langle a, b \rangle = \sum_{n=0}^{\infty} a_n \bar{b}_n, \qquad a \in l^2(\rho) \text{ and } b \in l^2(\tau).$$

The adjoint of the forward shift S on $l^2(\rho)$ is the backward shift B on $l^2(\tau)$ given by $B(b_0, b_1, \cdots)=(b_1, b_2, \cdots)$.

The mapping that sends a sequence b in $l^2(\tau)$ to the analytic function $\sum_{n=0}^{\infty} b_n z^n$ in the unit disk establishes an isometric isomorphism between

$l^2(\tau)$ and the space $H^2(\tau)$ consisting of all analytic functions in the Hardy class H^2 whose Fourier coefficients are in $l^2(\tau)$. We shall use the spaces $l^2(\tau)$ and $H^2(\tau)$ interchangeably. Let P denote the orthogonal projection of $L^2([0, 2\pi))$ onto H^2 (regarded as a subspace of L^2). For an analytic function φ in H^∞ we use the notation $\tilde{\varphi}$ to denote the analytic function in H^∞ whose nth Taylor coefficient is the complex conjugate of that of φ.

We shall now quote some relevant facts from [7]. (These are contained in the proof of Theorem 1.1 of [7].)

(I) For φ in H^∞ and g in $H^2(\tau)$ define $T_{\bar\varphi}g = P(\bar\varphi g)$. Then $T_{\bar\varphi}$ is a bounded operator from $H^2(\tau)$ into itself and $\|T_{\bar\varphi}\| \leqq \|\varphi\|_\infty$.

(II) If φ is a polynomial, $T_{\bar\varphi} = \tilde\varphi(B)$.

(III) If $\{\varphi_n\}$ is a sequence of polynomials and φ is an element of H^∞ such that $\{\|\varphi_n\|_\infty\}$ is bounded and such that $\varphi_n(e^{i\theta}) \to \varphi(e^{i\theta})$ for a.e. θ in $[0, 2\pi)$, then for every g in $H^2(\tau)$ a subsequence of $\{T_{\bar\varphi_n}g\}$ converges weakly to $T_{\bar\varphi}g$.

The operators B and S are both completely nonunitary contractions (in fact, $B^n \to 0$ strongly as $n \to \infty$). So we can make use of the functional calculus of Sz.-Nagy and Foias [13, Chapter 3]. If the sequence $\{\varphi_n\}$ in H^∞ and the element φ in H^∞ are such that $\{\|\varphi_n\|_\infty\}$ is bounded and $\varphi_n(e^{i\theta}) \to \varphi(e^{i\theta})$ a.e. in $[0, 2\pi)$, $\varphi_n(B) \to \varphi(B)$ in the strong operator topology [13, Theorem 2.1, p. 114 of the English translation]. By the observations (II) and (III) above we have the conclusion $\tilde\varphi(B) = T_{\bar\varphi}$. By the general results about the functional calculus of Sz.-Nagy and Foias $\tilde\varphi(B)^* = \varphi(S)$. In particular, we have $M_\varphi = \varphi(S)$ for every polynomial φ. By a direct verification we can prove that $T_{\bar\varphi}^* = M_\varphi$. Thus $M_\varphi = \varphi(S)$. Once again by the functional calculus, $M_{\varphi_n} \to M_\varphi$ in the strong operator topology if a sequence $\{\varphi_n\}$ in H^∞ converges to φ in H^∞ boundedly and pointwise a.e. on the unit circle. This completes the proof of the Lemma.

PROOF OF THEOREM 1. Let e_0 be the sequence $(1, 0, 0, \cdots)$. It suffices to prove that e_0 can be approximated in $l^2(\rho)$ by a finite linear combination of right translates of a. Write f_a in the form φ/ψ where φ and ψ are in H^∞ and φ is outer. Then $\psi f_a = \varphi$. Let α and β be the sequences of Fourier coefficients of φ and ψ, respectively. Then $\alpha = \beta * a$. By Beurling's theorem and the fact that φ is outer, e_0 can be approximated by a finite linear combination of the right translates of α in the norm of the space l^2 (i.e. the space $l^2(\rho)$ with $\rho(n) = 1$ for all n). Since our weight sequence $\{\rho(n)\}$ is decreasing, the same approximation works in the space $l^2(\rho)$. Thus, given $\varepsilon > 0$, we can find a sequence γ with only finitely many nonzero entries such that

$$\|\beta * a * \gamma - e_0\| < \varepsilon$$

where the norm above is the norm of $l^2(\rho)$. To complete the proof of the

theorem, it suffices to find a sequence γ' with only finitely many nonzero entries such that

$$\|\beta * a * \gamma - \gamma' * a * \gamma\| < \varepsilon.$$

This can be done easily by appealing to the lemma. Choose a sequence $\{\psi_n\}$ of polynomials such that $\psi_n(e^{i\theta}) \to \psi(e^{i\theta})$ a.e. on $[0, 2\pi)$ and such that $\{\|\psi_n\|_\infty\}$ is bounded. We will simply choose γ' to be the sequence of coefficients of ψ_n for a sufficiently large n.

The following corollary arose from a conversation with Allen Shields. It has been proved by de Leeuw and Rudin [6] when the weights are all equal to one.

COROLLARY 1.   *Let $\{\rho(n)\}$ be a weight sequence satisfying the conditions of Theorem 1. If $a \in l^2(\rho)$ is such that f_a has positive real part, then a is cyclic for the forward shift S on $l^2(\rho)$.*

PROOF.   Since $\operatorname{Re} f_a > 0$, the same is true for $\operatorname{Re}(1/f_a)$. It follows that $1/f_a$ is in H^p for $0 < p < 1$. (See [4, Theorem 3.2, p. 34].) It is also well known that H^p is contained in N^+ [4, p. 26]. Now the corollary follows from Theorem 1.

By the same kind of argument we can prove a more general result. (See [8, Theorem 4.4.9, p. 77].)

COROLLARY 2.   *Let $\{\rho(n)\}$ be a weight sequence satisfying the condition of Theorem 1. If for an element a in $l^2(\rho)$ the function f_a has the form g^k, where k is a positive integer and g is an analytic function in the unit disk with positive real part, then a is a cyclic vector for the forward shift S on $l^2(\rho)$.*

3. A problem of Aharonov, Shapiro and Shields.
After an earlier version of this paper was written, a preprint of the paper [1] by Aharonov, Shapiro and Shields was made available to the author. In that paper the authors study weakly invertible elements in the Bergman space of square-summable analytic functions in the unit disk. In the notations and terminology of the present paper, this is the study of cyclic vectors for the forward shift in the space $l^2(\rho)$ with $\rho(n) = (1+n)^{-1}$. We shall denote this space by A_1. More generally, let A_ε denote the space $l^2(\rho)$ with $\rho(n) = (1+n)^{-\varepsilon}$, where $\varepsilon > 0$. On the basis of the main result of [1] and earlier results of Shapiro, the following question was raised in [1].

QUESTION.   *Let a be a sequence in A_ε such that there exist positive constants c and k (depending on a) such that the inequality*

$$(6) \qquad\qquad |f_a(z)| \geqq c(1 - |z|)^k$$

holds in $\{z : |z| < 1\}$. Does it follow that a is a cyclic vector for the forward shift in A_ε?

Theorem 2 stated below can be used to give a partial answer to this question. This theorem can be proved by essentially the same argument used to prove Theorem 1.

THEOREM 2. *Let ρ be a weight sequence satisfying the conditions of Theorem 1. Let a be a vector in $l^2(\rho)$ such that $f_a = gf_b$, where $g^{-1} \in N^+$ and b is a cyclic vector for the forward shift on $l^2(\rho)$. Then a is cyclic for the forward shift on $l^2(\rho)$.*

THEOREM 3. *Let a be an element of A_ε ($\varepsilon > 0$) such that f_a is in the Hardy class H^p for some $p > 0$. If f_a satisfies (6), then a is a cyclic vector for the forward shift on A_ε.*

PROOF. f_a can be factored uniquely in the form $f_a = uF$ where u is inner and F is in H^p and is outer [4, p. 24]. It can be seen easily that F^{-1} is in the class N^+. Since F is H^p, there exists a constant M such that $|F(z)| \leq M(1 - |z|)^{-1/p}$ (see [4, p. 36]). Combining this with the assumption that f_a satisfies an inequality of the type (6), we see that u itself satisfies an inequality like (6). By Lemma 4 of [9] this implies that the sequence of Fourier coefficients of u is cyclic in A_ε. Now Theorem 3 follows from Theorem 2.

REMARK. For the case $p \geq 1$, Theorem 3 has been obtained earlier by Shapiro [11].

BIBLIOGRAPHY

1. D. Aharonov, H. S. Shapiro and A. L. Shields, *Weakly invertible elements in the space of square-summable holomorphic functions* (to appear).

2. A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta. Math. **81** (1948), 17pp. MR **10**, 381.

3. ———, *A critical topology in harmonic analysis on semigroups*, Acta Math. **112** (1964), 215–228. MR **29** #6255.

4. P. L. Duren, *Theory of H^p spaces*, Pure and Appl. Math., vol. 38, Academic Press, New York, 1970. MR **42** #3552.

5. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N.J., 1967. MR **34** #8178.

6. K. deLeeuw and W. Rudin, *Extreme points and extremum problems in H_1*, Pacific J. Math. **8** (1958), 467–485. MR **20** #5426.

7. M. Rabindranathan, *Toeplitz operators and division by inner functions*, Indiana Univ. Math. J. **22** (1972), 523–529.

8. W. Rudin, *Function theory in polydisks*, Benjamin, New York, 1969. MR **41** #501.

9. H. S. Shapiro, *Weakly invertible elements in certain function spaces, and generators in l_1*, Michigan Math. J. **11** (1964), 161–165. MR **29** #3620.

10. ———, *Weighted polynomial approximation and boundary behaviour of analytic functions*, Contemporary Problems in Theory Anal. Functions (Internat. Conf., Erevan, 1965), "Nauka," Moscow, 1966, pp. 326–335. MR **35** #383.

11. ———, *Some remarks on weighted polynomial approximations of holomorphic functions*, Mat. Sb. **73** (115) (1967), 320–330=Math. USSR-Sb. **2** (1967), 285–294. MR **36** #395.

12. A. L. Shields and L. J. Wallen, *The commutants of certain Hilbert space operators*, Indiana Univ. Math. J. **20** (1970–71), 777–788. MR **44** #4558.

13. B. Sz.-Nagy and C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson, Paris; Akad. Kiadó, Budapest, 1967; English transl., North-Holland, Amsterdam; American Elsevier, New York; Akad. Kiadó, Budapest, 1970. MR **37** #778; **43** #947.

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