PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

THE SILVERMAN NECESSARY CONDITION FOR MULTIPLE INTEGRALS IN THE CALCULUS OF VARIATIONS

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ABSTRACT. A necessary condition for a weak relative minimum of a multiple integral in the calculus of variations, recently derived by Silverman, is shown to be a consequence of (and hence equivalent to) the well-known Legendre-Hadamard necessary condition in the special cases of a double, triple or quadruple integral.

The Legendre-Hadamard condition [1, p. 11] is that

$$Q(\xi, \lambda) \equiv f_{ij}^{\alpha\beta}\lambda_\alpha\lambda_\beta\xi^i\xi^j \geqq 0 \tag{1}$$

for all $\xi=(\xi^1, \xi^2, \cdots, \xi^N)$ and $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_\nu)$. (We employ the usual summation convention involving repeated indices. Greek indices range from 1 to ν and Latin indices from 1 to N. The relation of $f_{ij}^{\alpha\beta}$ to the integrand of a multiple integral (with ν independent variables), whose minimum value is sought, is known [1, p. 11] but immaterial to this paper.) The Silverman condition [2] is that

$$Q'(\Xi, \Lambda) = f_{ij}^{\alpha\beta}(\lambda_\alpha^\gamma\lambda_\beta^\gamma\xi_\varepsilon^i\xi_\varepsilon^j + \lambda_\alpha^\gamma\lambda_\beta^\varepsilon\xi_\gamma^i\xi_\varepsilon^j + \lambda_\alpha^\gamma\lambda_\beta^\varepsilon\xi_\varepsilon^i\xi_\gamma^j) \geqq 0 \tag{2}$$

for all $\Xi=(\xi_\varepsilon^i)$ and $\Lambda=(\lambda_\alpha^\gamma)$.

Silverman showed that his condition implies the Legendre-Hadamard condition, but left the converse open. It is, however, an immediate consequence of the easily verified identity,

$$\begin{aligned} 2Q'(\Xi, \Lambda) &= 2(4-\nu)\sum_\gamma Q(\xi_\gamma, \lambda^\gamma) \\ &\quad + \sum_{\gamma<\varepsilon}\{Q(\xi_\gamma+\xi_\varepsilon, \lambda^\gamma+\lambda^\varepsilon) + Q(\xi_\gamma-\xi_\varepsilon, \lambda^\gamma-\lambda^\varepsilon)\}, \end{aligned} \tag{3}$$

in which $\xi_\gamma=(\xi_\gamma^1, \xi_\gamma^2, \cdots, \xi_\gamma^N)$, $\lambda^\gamma=(\lambda_1^\gamma, \lambda_2^\gamma, \cdots, \lambda_\nu^\gamma)$, that the converse is true, and hence the Silverman and Legendre-Hadamard conditions are equivalent, when $\nu\leqq 4$. The situation when $\nu\geqq 5$ remains open.

Received by the editors September 13, 1973.

AMS (MOS) subject classifications (1970). Primary 49B20; Secondary 15A63, 26A86.

Key words and phrases. Legendre-Hadamard necessary condition.

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