

## THE SILVERMAN NECESSARY CONDITION FOR MULTIPLE INTEGRALS IN THE CALCULUS OF VARIATIONS

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ABSTRACT. A necessary condition for a weak relative minimum of a multiple integral in the calculus of variations, recently derived by Silverman, is shown to be a consequence of (and hence equivalent to) the well-known Legendre-Hadamard necessary condition in the special cases of a double, triple or quadruple integral.

The Legendre-Hadamard condition [1, p. 11] is that

$$(1) \quad Q(\xi, \lambda) \equiv f_{i_j}^{\alpha\beta} \lambda_\alpha \lambda_\beta \xi^i \xi^j \geq 0$$

for all  $\xi = (\xi^1, \xi^2, \dots, \xi^N)$  and  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\nu)$ . (We employ the usual summation convention involving repeated indices. Greek indices range from 1 to  $\nu$  and Latin indices from 1 to  $N$ . The relation of  $f_{i_j}^{\alpha\beta}$  to the integrand of a multiple integral (with  $\nu$  independent variables), whose minimum value is sought, is known [1, p. 11] but immaterial to this paper.) The Silverman condition [2] is that

$$(2) \quad Q'(\Xi, \Lambda) = f_{i_j}^{\alpha\beta} (\lambda_\alpha^\gamma \lambda_\beta^\epsilon \xi^i \xi^j + \lambda_\alpha^\gamma \lambda_\beta^\epsilon \xi^i \xi^j + \lambda_\alpha^\gamma \lambda_\beta^\epsilon \xi^i \xi^j) \geq 0$$

for all  $\Xi = (\xi_\epsilon^i)$  and  $\Lambda = (\lambda_\epsilon^\gamma)$ .

Silverman showed that his condition implies the Legendre-Hadamard condition, but left the converse open. It is, however, an immediate consequence of the easily verified identity,

$$(3) \quad 2Q'(\Xi, \Lambda) = 2(4 - \nu) \sum_\gamma Q(\xi_\gamma, \lambda^\gamma) + \sum_{\gamma < \epsilon} \{Q(\xi_\gamma + \xi_\epsilon, \lambda^\gamma + \lambda^\epsilon) + Q(\xi_\gamma - \xi_\epsilon, \lambda^\gamma - \lambda^\epsilon)\},$$

in which  $\xi_\gamma = (\xi_\gamma^1, \xi_\gamma^2, \dots, \xi_\gamma^N)$ ,  $\lambda^\gamma = (\lambda_\gamma^1, \lambda_\gamma^2, \dots, \lambda_\gamma^\nu)$ , that the converse is true, and hence the Silverman and Legendre-Hadamard conditions are equivalent, when  $\nu \leq 4$ . The situation when  $\nu \geq 5$  remains open.

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## REFERENCES

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