ON THE SERIAL COMPLETION OF DELETED
SCHAUDER BASES BY DOMAIN ADJUSTMENT

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Abstract. Given a Schauder basis in a Banach function space of a specified type, one can delete any finite number of elements of the basis and still preserve serial totality by making an arbitrarily small adjustment of the domain.

Let \( \{\phi_n\}_{n=1}^\infty \) be a system of real valued functions finite almost everywhere and measurable on a set \( G \subseteq [0, 1] \), mes. \( G > 0 \). Talalyan [2], [3] proved the following to be equivalent:

(a) \( \{\phi_n\}_{n=1}^\infty \) is total in measure on \( G \), that is, for every measurable function \( f \) defined on \( G \), there exists a sequence of finite linear combinations of functions of the system \( \{\phi_n\}_{n=1}^\infty \) which converges in measure to \( f \) on \( G \).

(b) For each positive number \( \varepsilon \), there is a measurable set \( S_\varepsilon \) whose measure exceeds \( 1 - \varepsilon \), such that \( \{\phi_n\}_{n=1}^\infty \) is total in \( L^2(S_\varepsilon) \).

We shall say that \( \{\phi_k\}_{k=1}^\infty \) is serially total in some function space, if for any given function \( f \) in the space we can find a series \( \sum_{k=1}^\infty a_k \phi_k \) which converges to \( f \) in the metric of the space. The result of this paper can be viewed as a first step in changing total into serially total.

Let \( L(E) \) be a Banach space of measurable functions on a measurable set \( E \subseteq [a, b] \) with natural linear operations. As usual, identify functions equal almost everywhere. Postulate the following on \( L(E) \):

(1) \( L(E) \) is contained in \( L^1(E) \);
(2) \( L(E) \) contains the function 1;
(3) if \( f \in L(E) \), and if for a measurable function \( g \), \( 0 \leq g(x) \leq f(x) \) almost everywhere, then \( g \in L(E) \);
(4) if \( f \in L(E) \) and \( \chi_A \) is the characteristic function of the measurable set \( A \), then \( \|f \chi_A\| \leq \|f\|_1 \) goes to zero as \( |A| \) goes to zero, where \( |A| \) denotes the Lebesgue measure of \( A \).

Example of spaces that satisfy (1)–(4) are the \( L^p \) spaces \( 1 \leq p < \infty \) and the separable Orlicz spaces.

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Theorem. Let \( \{ \phi_k \}_{k=1}^{\infty} \) be a normalized basis for \( L(E) \), then given any natural number \( N_0 \) and \( \varepsilon > 0 \), there exists a set \( D = D(N_0, \varepsilon) \), contained in \( E \) and satisfying \( |D| > |E| - \varepsilon \), such that \( \{ \phi_k \}_{k=N_0}^{\infty} \) is serially total in \( L(D) \).

We should note that by [1] \( \{ \phi_k \}_{k=N_0}^{\infty} \) is serially total in measure on \( E \). We utilize a lemma from [1] as the main tool in the proof of the theorem.

Lemma. Let \( \{ \phi_k \}_{k=1}^{\infty} \) be a normalized Schauder basis for \( L(E) \), \( g \) a measurable function finite almost everywhere on \( E \). Then given \( \varepsilon > 0 \) and a natural number \( N \), there exists a set \( e_0 \) and real numbers \( b_{N+1}, \ldots, b_m \) such that

\[
\begin{align*}
& e_0 \subset E, \text{ and } |e_0| < \varepsilon; \\
& |b_k| < \varepsilon \quad \text{for } N + 1 \leq k \leq m; \\
& \left\| \sum_{k=N+1}^{m} b_k \phi_k - g \right\|_{L(E|e_0)} < \varepsilon; \\
& \left\| \sum_{k=N+1}^{s} b_k \phi_k \right\|_e \leq \varepsilon + \|g\|_e \quad \text{for all } N + 1 \leq s \leq m,
\end{align*}
\]

and every measurable subset \( e \) of \( E \setminus e_0 \).

Proof of Theorem. The required set \( D \) will be a certain infinite intersection. The individual factors of this intersection are inductively determined.

Choose a sequence of positive terms \( \varepsilon_n \) with the property

\[
\sum_{n=1}^{\infty} \varepsilon_n = \varepsilon.
\]

By virtue of the lemma we may choose a set \( D_1 \) whose complement \( E_1 \) has measure less than \( \varepsilon_1 \), and a \( \Phi \)-polynomial

\[
P_{11} = \sum_{j=v(1,0)+1}^{v(1,1)} b_j \phi_j \quad \text{where } v(1, 0) = N_0
\]

satisfying the following conditions:

\[
\begin{align*}
(5) & \quad |b_j| < \varepsilon_1, \quad \text{for } v(1, 0) < j \leq v(1, 1), \\
(6) & \quad \|\phi_1 - P_{11}\|_{E_1} < \varepsilon_1, \\
(7) & \quad \sup_{s \leq v(1,1)} \left\| \sum_{j=v(1,0)+1}^{s} b_j \phi_j \right\|_{E} < \varepsilon_1 + \|\phi_1\|_e
\end{align*}
\]

for all measurable subsets \( e \) contained in \( D_1 \).

Again applying the lemma twice in succession allows us to choose for
$i=1, 2$, sets $D_{2i}$ with respective complements $E_{2i}$ and $\Phi$-polynomials

$$P_{2i} = \sum_{j=v(2,i-1)+1}^{v(2,i)} b_j \phi_j$$

with $v(1, 1)<v(2, 0)<v(2, 1)<v(2, 2)$ such that

(8) $|E_{2i}| < \varepsilon_2/2$ for $i = 1, 2$;

(9) $|b_j| < \varepsilon_2$ if $v(2, 0) < j \leq v(2, 2)$;

(10) $\|\phi_1 - P_{11}\|_{D_{21}} < \varepsilon_2/2$;

(11) $\|\phi_2 - P_{22}\|_{D_{22}} < \varepsilon_2/2$;

(12) $\sup_{s \leq v(2, 1)} \left\| \sum_{j=v(2,0)+1}^{s} b_j \phi_j \right\|_e < \varepsilon_2 + \|\phi_1 - P_{11}\|_e$

for all measurable subsets $e$ of $D_{21}$;

(13) $\sup_{s \leq v(2, 2)} \left\| \sum_{j=v(2,1)+1}^{s} b_j \phi_j \right\|_e < \varepsilon_2 + \|\phi_2\|_e$

for all measurable subsets $e$ of $D_{22}$.

Let $D_2^* = \cap_{i=1}^{2} D_{2i}$ and $D_2 = D_1 \cap D_2^*$, then $|D_2| > |E| - \sum_{i=1}^{2} \varepsilon_i$. By virtue of (12), (13), (6) and the definition of the set $D_2$ we obtain

(14) $\sup_{s \leq v(2,k)} \left\| \sum_{j=v(2,k-1)+1}^{s} b_j \phi_j \right\|_e \leq \begin{cases} \varepsilon_2 + \varepsilon_1 & \text{if } k = 1; \\ \varepsilon_2 + 1 & \text{if } k = 2; \end{cases}$

for all measurable subsets $e$ of $D_2$.

In the $n$th step we apply the lemma successively to the functions

(15) $\Psi_k = \phi_k - \sum_{j=k}^{n-1} P_{jk}$ with $k = 1, 2, \ldots, n - 1$;

(16) $\Psi_n = \phi_n$.

The lemma makes it possible to choose for $k=1, 2, \ldots, n$, sets $D_{nk}$ with respective complements $E_{nk}$ and $\Phi$-polynomials

$$P_{nk} = \sum_{j=v(n,k-1)+1}^{v(n,k)} b_j \phi_j$$

where $v(n-1, n-1)<v(n, 0)<v(n, 1)<\cdots<v(n, n)$ such that the following holds:

(18) $|E_{nk}| < \varepsilon_n/n$;

(19) $|b_j| < \varepsilon_n$, for $v(n, 0) < j \leq v(n, n)$;

(20) $\|\Psi_k - P_{nk}\|_{D_{nk}} < \varepsilon_n/n$;

(21) $\sup_{s \leq v(n,k)} \left\| \sum_{j=v(n,k-1)+1}^{s} b_j \phi_j \right\|_e \leq \varepsilon_n + \|\Psi_k\|_e$
for all measurable subsets $e$ of $D_n$. Let $D_n^* = \bigcap_{k=1}^n D_{nk}$ and $D_n = D_{n-1} \cap D_n^*$, then $|D_n| > |E| - \sum_{k=1}^n \varepsilon_k$. In analogous fashion to (14) of the second step we obtain in the $n$th step

$$\sup_{\varepsilon \subseteq \nu(n,k)} \left\| \sum_{j=\nu(n,k-1)+1}^{\varepsilon} b_j \phi_j \right\| \leq \begin{cases} \varepsilon_n + \varepsilon_{n-1} & \text{if } k < n; \\ \varepsilon_n + 1 & \text{if } k = n; \end{cases}$$

for all measurable subsets $e$ of $D_n$. Continuing by inductive construction it is easy to see that (18)--(22) holds for each natural number $n$. Define $D = \bigcap_{n=1}^\infty D_n$, then $|D| \geq |E| - \sum_{k=1}^\infty \varepsilon_k \geq |E| - \varepsilon$.

Now we are ready to show that given any function $f$ in $F(\varepsilon)$ we can find a series from $\{\phi_j : v(m, 0) \leq j \leq v(m, m), \; m = 1, 2, \ldots\}$ which will converge to $f$ in the $L(D)$ norm. In fact, if $\sum_{k=1}^\infty a_k \phi_k$ is the Schauder basis expansion of $f$ then $\sum_{j=1}^n \sum_{k=1}^j a_k P_{jk}$ converges to $f$ in the $L(D)$ norm.

Let $\delta > 0$ be given. Choose $N_1$ so that

$$\left\| \sum_{k=1}^n a_k \phi_k - f \right\|_D \leq \left\| \sum_{k=1}^n a_k \phi_k - f \chi_D \right\| < \frac{\delta}{3} \quad \text{for all } n > N_1.$$

Setting $a = \sup_k |a_k|$, choose $N_2 > N_1$ so that $a \cdot \varepsilon_n < \delta/3$ for all $n > N_2$. By virtue of (20), (15) and the definition of $D$ we obtain

$$\left\| \sum_{j=1}^n \sum_{k=1}^j a_k P_{jk} - \sum_{k=1}^n a_k \phi_k \right\|_D \leq \sum_{k=1}^n \left\| \sum_{j=k}^n P_{jk} - \phi_k \right\|_D$$

$$\leq n \cdot a \cdot \varepsilon_n / n < \delta/3.$$

Last, choose $N_3 > N_2$ so that

$$|2 \cdot a_n| < \delta/3 \quad \text{whenever } n > N_3.$$

By virtue of (23) and (24) we obtain

$$\left\| \sum_{j=1}^n \sum_{k=1}^j a_k P_{jk} - f \right\|_D < \frac{2\delta}{3}, \quad \text{for all } n > N_3.$$

Obviously

$$\left\| \sum_{j=1}^n \sum_{k=1}^j a_k P_{jk} + \sum_{k=1}^{n+1} a_k P_{n+1k} - f \right\|_D < \frac{2\delta}{3}, \quad \text{for all } n > N_3.$$

If we add in only part of the second sum; that is,

$$\sum_{k=1}^m a_k P_{n+1k} \quad \text{with } m < n + 1$$
then it is easy to see from (20) that the basis elements $\phi_i, i=1, 2, \cdots, m$, will be approximated better than before, by $\varepsilon_{n+1}/(n+1)$ instead of $\varepsilon_n/n$. Hence via the calculations in (24), and by (26) it is immediate that

\begin{equation}
\left\| \sum_{j=1}^{n} \sum_{k=1}^{j} a_k P_{jk} + \sum_{k=1}^{m} a_k P_{n+1} - f \right\|_D < \frac{2\delta}{3}.
\end{equation}

Finally, if we add to the summations in (28) only part of the $\Phi$-polynomial $a_{m+1} P_{n+1} m+1$, let us say

$$\sum_{j=v(n+1, m)+1}^{s} a_{m+1} b_j \phi_j \quad \text{where} \quad v(n+1, m) < s < v(n+1, m+1)$$

then (22) and (25) in addition to (28) give us

$$\left\| \sum_{j=1}^{n} \sum_{k=1}^{j} a_k P_{jk} + \sum_{k=1}^{m} a_k P_{n+1} k + \sum_{j=v(n+1, m)+1}^{s} a_{n+1} b_j \phi_j \right\|_D < \delta.$$ 

Thus, we obtain the desired series convergence. Furthermore, the coefficients of $\phi_j$ go to zero, since the $a_n$ are bounded by $a$ and the $b_j$ go to zero.

**Bibliography**

