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ON THE SERIAL COMPLETION OF DELETED SCHAUDER BASES BY DOMAIN ADJUSTMENT

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ABSTRACT. Given a Schauder basis in a Banach function space of a specified type, one can delete any finite number of elements of the basis and still preserve serial totality by making an arbitrarily small adjustment of the domain.

Let $\{\phi_n\}_{n=1}^{\infty}$ be a system of real valued functions finite almost everywhere and measurable on a set $G\subset[0, 1]$, mes. $G>0$. Talalyan **[2]**, **[3]** proved the following to be equivalent:

(a) $\{\phi_n\}_{n=1}^{\infty}$ is total in measure on G, that is, for every measurable function f defined on G, there exists a sequence of finite linear combinations of functions of the system $\{\phi_n\}_{n=1}^{\infty}$ which converges in measure to f on G.

(b) For each positive number ε, there is a measurable set S_ε whose measure exceeds $1-\varepsilon$, such that $\{\phi_n\}_{n=1}^{\infty}$ is total in $L^2(S_\varepsilon)$.

We shall say that $\{\phi_k\}_{k=1}^{\infty}$ is serially total in some function space, if for any given function f in the space we can find a series $\sum_{k=1}^{\infty} a_k\phi_k$ which converges to f in the metric of the space. The result of this paper can be viewed as a first step in changing total into serially total.

Let $L(E)$ be a Banach space of measurable functions on a measurable set $E\subset[a, b]$ with natural linear operations. As usual, identify functions equal almost everywhere. Postulate the following on $L(E)$:

(1) *$L(E)$ is contained in $L^1(E)$;*

(2) *$L(E)$ contains the function* 1;

(3) *if $f\in L(E)$, and if for a measurable function g, $0\leqq g(x)\leqq f(x)$ almost everywhere, then $g\in L(E)$;*

(4) *if $f\in L(E)$ and χ_A is the characteristic function of the measurable set A, then $\|f\chi_A\|\equiv\|f\|_A$ goes to zero as $|A|$ goes to zero, where $|A|$ denotes the Lebesgue measure of A.*

Example of spaces that satisfy (1)–(4) are the L^p spaces $1\leqq p<\infty$ and the separable Orlicz spaces.

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THEOREM. *Let $\{\phi_k\}_{k=1}^{\infty}$ be a normalized basis for $L(E)$, then given any natural number N_0 and $\varepsilon>0$, there exists a set $D=D(N_0, \varepsilon)$, contained in E and satisfying $|D|>|E|-\varepsilon$, such that $\{\phi_k\}_{k=N_0}^{\infty}$ is serially total in $L(D)$.*

We should note that by [1] $\{\phi_k\}_{k=N_0}^{\infty}$ is serially total in measure on E. We utilize a lemma from [1] as the main tool in the proof of the theorem.

LEMMA. *Let $\{\phi_k\}_{k=1}^{\infty}$ be a normalized Schauder basis for $L(E)$, g a measurable function finite almost everywhere on E. Then given $\varepsilon>0$ and a natural number N, there exists a set e_0 and real numbers $b_{N+1}, \cdots, b_m$ such that*

$$e_0 \subset E, \textit{ and } |e_0| < \varepsilon;$$

$$|b_k| < \varepsilon \quad \textit{for } N+1 \leqq k \leqq m;$$

$$\left\| \sum_{k=N+1}^{m} b_k\phi_k - g \right\|_{(E\setminus e_0)} < \varepsilon;$$

$$\left\| \sum_{k=N+1}^{s} b_k\phi_k \right\|_{e} \leqq \varepsilon + \|g\|_e \quad \textit{for all } N+1 \leqq s \leqq m,$$

and every measurable subset e of $E\setminus e_0$.

PROOF OF THEOREM. The required set D will be a certain infinite intersection. The individual factors of this intersection are inductively determined.

Choose a sequence of positive terms ε_n with the property

$$\sum_{n=1}^{\infty} \varepsilon_n = \varepsilon.$$

By virtue of the lemma we may choose a set D_1 whose complement E_1 has measure less than ε_1, and a Φ-polynomial

$$P_{11} = \sum_{j=\nu(1,0)+1}^{\nu(1,1)} b_j\phi_j \quad \text{where } \nu(1,0) = N_0$$

satisfying the following conditions:

$$|b_j| < \varepsilon_1, \quad \text{for } \nu(1,0) < j \leqq \nu(1,1), \tag{5}$$

$$\|\phi_1 - P_{11}\|_{D_1} < \varepsilon_1, \tag{6}$$

$$\sup_{s\leqq \nu(1,1)} \left\| \sum_{j=\nu(1,0)+1}^{s} b_j\phi_j \right\|_e < \varepsilon_1 + \|\phi_1\|_e \tag{7}$$

for all measurable subsets e contained in D_1.

Again applying the lemma twice in succession allows us to choose for

$i=1, 2$, sets D_{2i} with respective complements E_{2i} and Φ-polynomials

$$P_{2i} = \sum_{j=\nu(2,i-1)+1}^{\nu(2,i)} b_j\phi_j$$

with $\nu(1, 1)<\nu(2, 0)<\nu(2, 1)<\nu(2, 2)$ such that

$$|E_{2i}| < \varepsilon_2/2 \quad \text{for } i = 1, 2; \tag{8}$$

$$|b_j| < \varepsilon_2 \quad \text{if } \nu(2, 0) < j \leqq \nu(2, 2); \tag{9}$$

$$\|(\phi_1 - P_{11}) - P_{21}\|_{D_{21}} < \varepsilon_2/2; \tag{10}$$

$$\|\phi_2 - P_{22}\|_{D_{22}} < \varepsilon_2/2; \tag{11}$$

$$\sup_{s\leqq\nu(2,1)} \left\| \sum_{j=\nu(2,0)+1}^{s} b_j\phi_j \right\|_e < \varepsilon_2 + \|\phi_1 - P_{11}\|_e \tag{12}$$

for all measurable subsets e of D_{21};

$$\sup_{s\leqq\nu(2,2)} \left\| \sum_{j=\nu(2,1)+1}^{s} b_j\phi_j \right\|_e < \varepsilon_2 + \|\phi_2\|_e \tag{13}$$

for all measurable subsets e of D_{22}.

Let $D_2^*=\bigcap_{i=1}^2 D_{2i}$ and $D_2=D_1\cap D_2^*$, then $|D_2|>|E|-\sum_{i=1}^2 \varepsilon_1$. By virtue of (12), (13), (6) and the definition of the set D_2 we obtain

$$\sup_{s\leqq\nu(2,k)} \left\| \sum_{j=\nu(2,k-1)+1}^{s} b_j\phi_j \right\|_e \leqq \begin{cases} \varepsilon_2 + \varepsilon_1 & \text{if } k = 1; \\ \varepsilon_2 + 1 & \text{if } k = 2; \end{cases} \tag{14}$$

for all measurable subsets e of D_2.

In the nth step we apply the lemma successively to the functions

$$\Psi_k = \phi_k - \sum_{j=k}^{n-1} P_{jk} \quad \text{with } k = 1, 2, \cdots, n-1, \tag{15}$$

$$\Psi_n = \phi_n. \tag{16}$$

The lemma makes it possible to choose for $k=1, 2, \cdots, n$, sets D_{nk} with respective complements E_{nk} and Φ-polynomials

$$P_{nk} = \sum_{j=\nu(n,k-1)+1}^{\nu(n,k)} b_j\phi_j \tag{17}$$

where $\nu(n-1, n-1)<\nu(n, 0)<\nu(n, 1)<\cdots<\nu(n, n)$ such that the following holds:

$$|E_{nk}| < \varepsilon_n/n; \tag{18}$$

$$|b_j| < \varepsilon_n, \quad \text{for } \nu(n, 0) < j \leqq \nu(n, n); \tag{19}$$

$$\|\Psi_k - P_{nk}\|_{D_{nk}} < \varepsilon_n/n; \tag{20}$$

$$\sup_{s\leqq\nu(n,k)} \left\| \sum_{j=\nu(n,k-1)+1}^{s} b_j\phi_j \right\|_e \leqq \varepsilon_n + \|\Psi_k\|_e \tag{21}$$

for all measurable subsets e of D_{nk}. Let $D_n^* = \bigcap_{k=1}^n D_{nk}$ and $D_n = D_{n-1} \cap D_n^*$, then $|D_n| > |E| - \sum_{k=1}^n \varepsilon_k$. In analogous fashion to (14) of the second step we obtain in the nth step

$$\sup_{s \leqq \nu(n,k)} \left\| \sum_{j=\nu(n,k-1)+1}^{s} b_j \phi_j \right\|_e \leqq \begin{cases} \varepsilon_n + \varepsilon_{n-1} & \text{if } k < n; \\ \varepsilon_n + 1 & \text{if } k = n; \end{cases} \tag{22}$$

for all measurable subsets e of D_n. Continuing by inductive construction it is easy to see that (18)–(22) holds for each natural number n. Define $D = \bigcap_{n=1}^\infty D_n$, then $|D| \geqq |E| - \sum_{k=1}^\infty \varepsilon_k \geqq |E| - \varepsilon$.

Now we are ready to show that given any function f in $L(D)$ we can find a series from $\{\phi_j : \nu(m, 0) \leqq j \leqq \nu(m, m),\ m = 1, 2, \cdots\}$ which will converge to f in the $L(D)$ norm. In fact, if $\sum_{k=1}^\infty a_k \phi_k$ is the Schauder basis expansion of f then $\sum_{j=1}^\infty \sum_{k=1}^j a_k P_{jk}$ converges to f in the $L(D)$ norm.

Let $\delta > 0$ be given. Choose N_1 so that

$$\left\| \sum_{k=1}^n a_k \phi_k - f \right\|_D \leqq \left\| \sum_{k=1}^n a_k \phi_k - f \chi_D \right\| < \frac{\delta}{3} \quad \text{for all } n > N_1. \tag{23}$$

Setting $a = \sup_k |a_k|$, choose $N_2 > N_1$ so that $a \cdot \varepsilon_n < \delta/3$ for all $n > N_2$. By virtue of (20), (15) and the definition of D we obtain

$$\begin{aligned} \left\| \sum_{j=1}^n \sum_{k=1}^j a_k P_{jk} - \sum_{k=1}^n a_k \phi_k \right\|_D &= \left\| \sum_{k=1}^n a_k \left(\sum_{j=k}^n P_{jk} - \phi_k \right) \right\|_D \\ &\leqq \sum_{k=1}^n \left\| \sum_{j=k}^n P_{jk} - \phi_k \right\|_D \\ &\leqq n \cdot a \cdot \varepsilon_n / n < \delta/3 . \end{aligned} \tag{24}$$

Last, choose $N_3 > N_2$ so that

$$|2 \cdot a_n| < \delta/3 \quad \text{whenever } n > N_3. \tag{25}$$

By virtue of (23) and (24) we obtain

$$\left\| \sum_{j=1}^n \sum_{k=1}^j a_k P_{jk} - f \right\|_D < \frac{2\delta}{3}, \quad \text{for all } n > N_3. \tag{26}$$

Obviously

$$\left\| \sum_{j=1}^n \sum_{k=1}^j a_k P_{jk} + \sum_{k=1}^{n+1} a_k P_{n+1\,k} - f \right\|_D < \frac{2\delta}{3} \quad \text{for ail } n > N_3. \tag{27}$$

If we add in only part of the second sum; that is,

$$\sum_{k=1}^m a_k P_{n+1\,k} \quad \text{with } m < n + 1$$

then it is easy to see from (20) that the basis elements $\phi_i, i=1, 2, \cdots, m$, will be approximated better than before, by $\varepsilon_{n+1}/(n+1)$ instead of by ε_n/n. Hence via the calculations in (24), and by (26) it is immediate that

$$(28) \qquad \left\| \sum_{j=1}^{n} \sum_{k=1}^{j} a_k P_{jk} + \sum_{k=1}^{m} a_k P_{n+1\,k} - f \right\|_D < \frac{2\delta}{3}.$$

Finally, if we add to the summations in (28) only part of the Φ-polynomial $a_{m+1} P_{n+1\ m+1}$, let us say

$$\sum_{j=\nu(n+1,m)+1}^{s} a_{m+1} b_j \phi_j \quad \text{where } \nu(n+1, m) < s < \nu(n+1, m+1)$$

then (22) and (25) in addition to (28) give us

$$\left\| \sum_{j=1}^{n} \sum_{k=1}^{j} a_k P_{jk} + \sum_{k=1}^{m} a_k P_{n+1\,k} + \sum_{j=\nu(n+1,m)+1}^{s} a_{n+1} b_j \phi_j - f \right\|_D < \delta.$$

Thus, we obtain the desired series convergence. Furthermore, the coefficients of ϕ_j go to zero, since the a_n are bounded by a and the b_j go to zero.

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