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SETS THAT REMAIN MAXIMAL MONOTONE UNDER ALL MONOTONE PERTURBATIONS

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ABSTRACT. We characterize maximal monotone sets whose domain has nonempty interior and which remain maximal monotone under all perturbations.

Let H be a real Hilbert space and let A be a maximal monotone set with open domain $D(A)$. From [1] it follows that if B is maximal monotone such that $D(B)\cap D(A)\neq\varnothing$ then $A+B$ is maximal monotone; i.e., A remains maximal monotone under all maximal monotone perturbations B satisfying $D(A)\cap D(B)\neq\varnothing$.

At the Fourth C.I.M.E. Session, Varenna, 1970, A. Pazy raised the question of the characterization of such maximal monotone sets A. In the present note we answer Pazy's question for the special case of maximal monotone sets A satisfying $\operatorname{int}(D(A))\neq\varnothing$.

Let I be the set of maximal monotone A satisfying $\varnothing\neq\text{interior } D(A)\neq D(A)$. Let P be the set of sets A that remain maximal monotone under all perturbations. Let M be the set of maximal monotone sets A having the property that for x in $D(A)$ and N a supporting closed hyperplane to $D(A)$ at x, $P_N(Ax)=N$, where P_N is the orthogonal projection on N.

THEOREM. *$\varnothing\neq P\cap I=M\cap I\subsetneq I$. Also, $P\subset M$.*

PROOF. We show first that $M\cap I\subset P$. Suppose $A\in M\cap I$. Suppose B is maximal monotone and $D(B)\cap D(A)\neq\varnothing$. If $D(B)\cap\operatorname{int} D(A)\neq\varnothing$, then $A+B$ is maximal by [1]. If not, there exists x in $D(B)\cap D(A)-\operatorname{int} D(A)$. Since $\operatorname{cl}(D(A))$, $\operatorname{cl}(D(B))$ are convex and $\operatorname{int} D(A)\not\equiv\varnothing$ there is a closed hyperplane N separating $D(A)$ and $D(B)$. Take y in H such that $N=\{z\mid(z,y)=(x,y)\}$ and $(a,y)\leqq(x,y)$ for a in $D(A)$ and $(b,y)\geqq(x,y)$ for b in $D(B)$. Take $[x,x^*]$ in B. For $[b,b^*]$ in B, and $k\geqq 0$, $(x^*-b^*-ky, x-b)\geqq 0$. By the maximality of B, $Bx=\{Bx-ky:k\geqq 0\}$. Similarly, $Ax=\{Ax-ky:k\leqq 0\}$. Since $P_N(Ax)=N$, $Ax+Bx=H$. Hence, $A+B=\{[x,z]:z\in H\}$ giving $A+B$ maximal monotone, and proving $M\cap I\subset P$.

We now show $P\subset M$. We suppose A is maximal monotone but not in M.

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By translation we may suppose 0 is in $D(A)$, x_0 is in H, $D(A)\subset\{w|(w, x_0)\leqq 0\}$, and $P_NA(0)\neq N$ where $N=\{w|(w, x_0)=0\}$. Define B by $D(B)=\{z:(z, x_0)>0\}\cup\{0\}$, $B(0)=\{-kx_0:k\geqq 0\}$, and if $(z, x_0)>0$, $B(z)=z+(z, x_0)x_0-(z, z)(z, x_0)^{-1}x_0$. B is maximal monotone by Lemma 1 below, but $D(A+B)=\{0\}$ and $(A+B)(0)\subseteq\{A(0)+kx_0:k \text{ real}\}\neq H$ since $P_NA(0)\neq N$.

LEMMA 1. *B defined above is maximal monotone.*

PROOF. For z satisfying $(z, x_0)>0$, Bz may be described as follows. Consider a two-dimensional vector space containing x_0 and z, and suppose x_0 has coordinates $(1,0)$ and z has coordinates (x,y). Then $Bz=((x^2-y^2)/x, y)$. We show B is monotone. Suppose $(z_1, x_0)>0$ and $(z_2, x_0)>0$; then we take a three-dimensional vector space containing x_0, z_1 and z_2. In this space, R^3, for $z=(d, e, f)$ in R^3 $(d>0)$, we have $Bz=(d-(e^2+f^2)/d, e, f)$. If $z_1=(d_1, e_1, f_1)$ and $z_2=(d_2, e_2, f_2)$,

$$\begin{aligned}(Bz_2 - Bz_1, z_2 - z_1) &= (d_2 - d_1)^2 + (e_2 - e_1)^2 + (f_2 - f_1)^2 \\ &\qquad - (d_2 - d_1)((e_2^2 + f_2^2)/d_2 - (e_1^2 + f_1^2)/d_1) \\ &= (d_2 - d_1)^2 - 2e_1e_2 - 2f_1f_2 \\ &\qquad + (e_1^2 + f_1^2)d_2/d_1 + (e_2^2 + f_2^2)d_1/d_2.\end{aligned}$$

Using $2ab\leqq a^2+b^2$, where $a=e_1(d_2/d_1)^{1/2}$ and $b=e_2(d_1/d_2)^{1/2}$, etc., gives the right-hand side $\geqq(d_2-d_1)^2$. To complete the proof that B is monotone we take z_1 with $(x_0, z_1)>0$ and $z_2=0$.

$$\begin{aligned}&(Bz_2 - Bz_1, z_2 - z_1) \\ &\quad = (-kx_0 - z_1 - (z_1, x_0)x_0 + (z_1, z_1)(z_1, x_0)^{-1}x_1, -z_1), \qquad k > 0, \\ &\quad = k(x_0, z_1) + (z_1, x_0)^2.\end{aligned}$$

We show B is maximal. To show $R(1+B)=H$ it is enough to show that the operator B in R^2 described at the beginning of the lemma satisfies $R(1+B)=R^2$. Given (u, v) in R^2, if $v=0$, $u\leqq 0$, then $(I+B)(0, 0)$ contains (u, v), otherwise we take (x, y) satisfying $2x-y^2/x=u$, and $2y=v$. Q.E.D.

We show that $P\cap I\neq I$ by noting that B constructed in Lemma 1 is in I but not in P.

We show $P\cap I\neq\varnothing$ by the following result, completing the proof of the Theorem.

LEMMA 2. *Take x_0 in H, of norm 1, and for z satisfying $(x_0, z)>0$, let $Az=2(z, x_0)^{-1}z-(z, z)(z, x_0)^{-2}x_0-x_0$. Let $A(0)=\{z:4(z, x_0)+(z, z)-(z, x_0)^2\leqq 0\}$. Otherwise let $Az=\varnothing$. The A is in $M\cap I$.*

PROOF. In a two-dimensional space R^2 containing $x_0=(1,0)$ we have, if $z=(x,y)$, $x>0$, $Az=(-y^2/x^2, 2y/x)$, and $A(0)\cap R^2=\{(u,v):4u+v^2\leqq 0\}$. With $N=\{z:(z,x_0)=0\}$, the only hyperplane supporting $D(A)$, since $N\cap R^2$ is the set $\{(0,v):v \text{ in } R\}$, we have $P_N A(0)=N$.

We show A is monotone. Take z_1 and z_2 such that $(z_1,x_0)>0$ and $(z_2,x_0)>0$. Taking a three-dimensional vector space containing x_0, z_1, and z_2, we have, for $x_0=(1,0,0)$ and $z=(d,e,f)$, $d>0$, that $Az=(-(e^2+f^2)/d^2, 2e/d, 2f/d)$. If $z_1=(d_1,e_1,f_1)$ and $z_2=(d_2,e_2,f_2)$ then

$$\begin{aligned}(Az_2-Az_1, z_2-z_1) = {} & (-(e_2^2+f_2^2)/d_2^2+(e_1^2+f_1^2)/d_1^2)(d_2-d_1)\\ & +(e_2/d_2-e_1/d_1)(d_2-d_1)\\ & +(2e_2/d_2-2e_1/d_1)(e_2-e_1)\\ & +(2f_2/d_2-2f_1/d_1)(f_2-f_1).\end{aligned}$$

Using $2ab\leqq a^2+b^2$, where $a=e_1d_2^{1/2}/d_1$ and $b=e_2/d_2^{1/2}$, etc., gives the right-hand side $\geqq 0$. To complete the proof that A is monotone we consider z_1 with $(x_0,z_1)>0$, and $z_2=0$. In R^2, if $z_1=(x,y)$, then if $(u,v)\in A(0)$,

$$\begin{aligned}(Az_1-Az_2, z_1-z_2) &= (-y^2/x^2-u)x+(2y/x-v)y\\ &\geqq -x(u+v^2/4)\geqq 0,\end{aligned}$$

using $2ab\leqq a^2+b^2$ with $a=(x/2)^{1/2}v$ and $b=y(x/2)^{-1/2}$.

To show $R(1+A)=H$, as in the proof of Lemma 1 it suffices to show that for A in R^2, $R(I+A)=R^2$. Let $(u,v)\in R^2$. If $4u+v^2>0$ then there is a unique $x>0$ such that $u=x-v^2(2+x)^{-2}$. Choosing $y=vx(2+x)^{-1}$ we have $(u,v)=(I+A)(x,y)$. If $4u+v^2\leqq 0$ then $(u,v)\in(I+A)(0,0)$ and the proof is complete.

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