

## SETS THAT REMAIN MAXIMAL MONOTONE UNDER ALL MONOTONE PERTURBATIONS

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**ABSTRACT.** We characterize maximal monotone sets whose domain has nonempty interior and which remain maximal monotone under all perturbations.

Let  $H$  be a real Hilbert space and let  $A$  be a maximal monotone set with open domain  $D(A)$ . From [1] it follows that if  $B$  is maximal monotone such that  $D(B) \cap D(A) \neq \emptyset$  then  $A+B$  is maximal monotone; i.e.,  $A$  remains maximal monotone under all maximal monotone perturbations  $B$  satisfying  $D(A) \cap D(B) \neq \emptyset$ .

At the Fourth C.I.M.E. Session, Varenna, 1970, A. Pazy raised the question of the characterization of such maximal monotone sets  $A$ . In the present note we answer Pazy's question for the special case of maximal monotone sets  $A$  satisfying  $\text{int}(D(A)) \neq \emptyset$ .

Let  $I$  be the set of maximal monotone  $A$  satisfying  $\emptyset \neq \text{interior } D(A) \neq D(A)$ . Let  $P$  be the set of sets  $A$  that remain maximal monotone under all perturbations. Let  $M$  be the set of maximal monotone sets  $A$  having the property that for  $x$  in  $D(A)$  and  $N$  a supporting closed hyperplane to  $D(A)$  at  $x$ ,  $P_N(Ax) = N$ , where  $P_N$  is the orthogonal projection on  $N$ .

**THEOREM.**  $\emptyset \neq P \cap I = M \cap I \subseteq I$ . Also,  $P \subset M$ .

**PROOF.** We show first that  $M \cap I \subset P$ . Suppose  $A \in M \cap I$ . Suppose  $B$  is maximal monotone and  $D(B) \cap D(A) \neq \emptyset$ . If  $D(B) \cap \text{int } D(A) \neq \emptyset$ , then  $A+B$  is maximal by [1]. If not, there exists  $x$  in  $D(B) \cap D(A) - \text{int } D(A)$ . Since  $\text{cl}(D(A))$ ,  $\text{cl}(D(B))$  are convex and  $\text{int } D(A) \neq \emptyset$  there is a closed hyperplane  $N$  separating  $D(A)$  and  $D(B)$ . Take  $y$  in  $H$  such that  $N = \{z \mid (z, y) = (x, y)\}$  and  $(a, y) \leq (x, y)$  for  $a$  in  $D(A)$  and  $(b, y) \geq (x, y)$  for  $b$  in  $D(B)$ . Take  $[x, x^*]$  in  $B$ . For  $[b, b^*]$  in  $B$ , and  $k \geq 0$ ,  $(x^* - b^* - ky, x - b) \geq 0$ . By the maximality of  $B$ ,  $Bx = \{Bx - ky \mid k \geq 0\}$ . Similarly,  $Ax = \{Ax - ky \mid k \leq 0\}$ . Since  $P_N(Ax) = N$ ,  $Ax + Bx = H$ . Hence,  $A+B = \{[x, z] \mid z \in H\}$  giving  $A+B$  maximal monotone, and proving  $M \cap I \subset P$ .

We now show  $P \subset M$ . We suppose  $A$  is maximal monotone but not in  $M$ .

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By translation we may suppose 0 is in  $D(A)$ ,  $x_0$  is in  $H$ ,  $D(A) \subset \{w \mid (w, x_0) \leq 0\}$ , and  $P_{NA}(0) \neq N$  where  $N = \{w \mid (w, x_0) = 0\}$ . Define  $B$  by  $D(B) = \{z \mid (z, x_0) > 0\} \cup \{0\}$ ,  $B(0) = \{-kx_0 \mid k \geq 0\}$ , and if  $(z, x_0) > 0$ ,  $B(z) = z + (z, x_0)x_0 - (z, z)(z, x_0)^{-1}x_0$ .  $B$  is maximal monotone by Lemma 1 below, but  $D(A+B) = \{0\}$  and  $(A+B)(0) \subseteq \{A(0) + kx_0 \mid k \text{ real}\} \neq H$  since  $P_{NA}(0) \neq N$ .

LEMMA 1.  $B$  defined above is maximal monotone.

PROOF. For  $z$  satisfying  $(z, x_0) > 0$ ,  $Bz$  may be described as follows. Consider a two-dimensional vector space containing  $x_0$  and  $z$ , and suppose  $x_0$  has coordinates  $(1, 0)$  and  $z$  has coordinates  $(x, y)$ . Then  $Bz = ((x^2 - y^2)/x, y)$ . We show  $B$  is monotone. Suppose  $(z_1, x_0) > 0$  and  $(z_2, x_0) > 0$ ; then we take a three-dimensional vector space containing  $x_0, z_1$  and  $z_2$ . In this space,  $R^3$ , for  $z = (d, e, f)$  in  $R^3$  ( $d > 0$ ), we have  $Bz = (d - (e^2 + f^2)/d, e, f)$ . If  $z_1 = (d_1, e_1, f_1)$  and  $z_2 = (d_2, e_2, f_2)$ ,

$$\begin{aligned} (Bz_2 - Bz_1, z_2 - z_1) &= (d_2 - d_1)^2 + (e_2 - e_1)^2 + (f_2 - f_1)^2 \\ &\quad - (d_2 - d_1)((e_2^2 + f_2^2)/d_2 - (e_1^2 + f_1^2)/d_1) \\ &= (d_2 - d_1)^2 - 2e_1e_2 - 2f_1f_2 \\ &\quad + (e_1^2 + f_1^2)d_2/d_1 + (e_2^2 + f_2^2)d_1/d_2. \end{aligned}$$

Using  $2ab \leq a^2 + b^2$ , where  $a = e_1(d_2/d_1)^{1/2}$  and  $b = e_2(d_1/d_2)^{1/2}$ , etc., gives the right-hand side  $\geq (d_2 - d_1)^2$ . To complete the proof that  $B$  is monotone we take  $z_1$  with  $(x_0, z_1) > 0$  and  $z_2 = 0$ .

$$\begin{aligned} (Bz_2 - Bz_1, z_2 - z_1) &= (-kx_0 - z_1 - (z_1, x_0)x_0 + (z_1, z_1)(z_1, x_0)^{-1}x_1, -z_1), \quad k > 0, \\ &= k(x_0, z_1) + (z_1, x_0)^2. \end{aligned}$$

We show  $B$  is maximal. To show  $R(1+B) = H$  it is enough to show that the operator  $B$  in  $R^2$  described at the beginning of the lemma satisfies  $R(1+B) = R^2$ . Given  $(u, v)$  in  $R^2$ , if  $v = 0, u \leq 0$ , then  $(I+B)(0, 0)$  contains  $(u, v)$ , otherwise we take  $(x, y)$  satisfying  $2x - y^2/x = u$ , and  $2y = v$ . Q.E.D.

We show that  $P \cap I \neq I$  by noting that  $B$  constructed in Lemma 1 is in  $I$  but not in  $P$ .

We show  $P \cap I \neq \emptyset$  by the following result, completing the proof of the Theorem.

LEMMA 2. Take  $x_0$  in  $H$ , of norm 1, and for  $z$  satisfying  $(x_0, z) > 0$ , let  $Az = 2(z, x_0)^{-1}z - (z, z)(z, x_0)^{-2}x_0 - x_0$ . Let  $A(0) = \{z \mid 4(z, x_0) + (z, z) - (z, x_0)^2 \leq 0\}$ . Otherwise let  $Az = \emptyset$ . The  $A$  is in  $M \cap I$ .

PROOF. In a two-dimensional space  $R^2$  containing  $x_0=(1, 0)$  we have, if  $z=(x, y)$ ,  $x>0$ ,  $Az=(-y^2/x^2, 2y/x)$ , and  $A(0)\cap R^2=\{(u, v): 4u+v^2\leq 0\}$ . With  $N=\{z:(z, x_0)=0\}$ , the only hyperplane supporting  $D(A)$ , since  $N\cap R^2$  is the set  $\{(0, v):v \text{ in } R\}$ , we have  $P_{N^c}A(0)=N$ .

We show  $A$  is monotone. Take  $z_1$  and  $z_2$  such that  $(z_1, x_0)>0$  and  $(z_2, x_0)>0$ . Taking a three-dimensional vector space containing  $x_0, z_1$ , and  $z_2$ , we have, for  $x_0=(1, 0, 0)$  and  $z=(d, e, f)$ ,  $d>0$ , that  $Az=(-(e^2+f^2)/d^2, 2e/d, 2f/d)$ . If  $z_1=(d_1, e_1, f_1)$  and  $z_2=(d_2, e_2, f_2)$  then

$$\begin{aligned}(Az_2 - Az_1, z_2 - z_1) &= (-(e_2^2 + f_2^2)/d_2^2 + (e_1^2 + f_1^2)/d_1^2)(d_2 - d_1) \\ &\quad + (e_2/d_2 - e_1/d_1)(d_2 - d_1) \\ &\quad + (2e_2/d_2 - 2e_1/d_1)(e_2 - e_1) \\ &\quad + (2f_2/d_2 - 2f_1/d_1)(f_2 - f_1).\end{aligned}$$

Using  $2ab\leq a^2+b^2$ , where  $a=e_1d_2^{1/2}/d_1$  and  $b=e_2/d_2^{1/2}$ , etc., gives the right-hand side  $\geq 0$ . To complete the proof that  $A$  is monotone we consider  $z_1$  with  $(x_0, z_1)>0$ , and  $z_2=0$ . In  $R^2$ , if  $z_1=(x, y)$ , then if  $(u, v)\in A(0)$ ,

$$\begin{aligned}(Az_1 - Az_2, z_1 - z_2) &= (-y^2/x^2 - u)x + (2y/x - v)y \\ &\geq -x(u + v^2/4) \geq 0,\end{aligned}$$

using  $2ab\leq a^2+b^2$  with  $a=(x/2)^{1/2}v$  and  $b=y(x/2)^{-1/2}$ .

To show  $R(I+A)=H$ , as in the proof of Lemma 1 it suffices to show that for  $A$  in  $R^2$ ,  $R(I+A)=R^2$ . Let  $(u, v)\in R^2$ . If  $4u+v^2>0$  then there is a unique  $x>0$  such that  $u=x-v^2(2+x)^{-2}$ . Choosing  $y=vx(2+x)^{-1}$  we have  $(u, v)=(I+A)(x, y)$ . If  $4u+v^2\leq 0$  then  $(u, v)\in (I+A)(0, 0)$  and the proof is complete.

#### REFERENCE

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