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ON $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}$

MICHAEL CWIKEL

ABSTRACT. The Lions-Peetre formula for $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}$ valid for $q=p(\theta)$, where $1/p(\theta)=(1-\theta)/p_0+\theta/p_1$, is shown to have no reasonable generalization for any $q\neq p(\theta)$.

Let B be a Banach space, and (X, Σ, μ) a measure space. For $1\leqq p\leqq\infty$, $L_X^p(B)$ is defined as the Banach space of strongly measurable B-valued functions f on X, for which $\|f(x)\|_B$ belongs to the corresponding L^p space L_X^p of real valued functions. The subscript X is omitted where this would not cause any ambiguity. We also take X^2 to denote the measure space obtained by a cartesian product of X with itself, equipped with product measure, and R_+ to denote $(0, \infty)$ equipped with Lebesgue measure.

Let $(A_0, A_1, \mathscr{A})$ be an interpolation triple. It was shown in [3] that for p_0, p_1 in $[1, \infty]$ and $0<\theta<1$,

$$(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q} = L^q((A_0, A_1)_{\theta,q})$$

provided that $q=p(\theta)$, where $1/p(\theta)=(1-\theta)/p_0+\theta/p_1$. The problem of identifying $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}$ for other values of q was left open. In the special case where $A_0=A_1$, $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}=L(p(\theta), q)(A_0)$, the space of strongly measurable A_0-valued functions f such that $\|f(x)\|_{A_0}$ belongs to $L(p(\theta), q)$. (For definitions and details concerning the Lorentz spaces $L(p, q)$, see [1], [2], [4].) This example, as well as the Lions-Peetre formula, suggests that, given p_0, p_1, θ and q, it might be possible to find a normed space A, $A\subset A_0+A_1$, such that the membership of $f(x)$ in $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}$ is determined solely by the behaviour of the real valued function $\|f(x)\|_A$.

What is such a space A likely to be? A rather natural guess would be $A=(A_0, A_1)_{\theta,q}$, and indeed, if the measure space consists of a single atom of finite measure,

$$\begin{aligned}(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q} &= (A_0, A_1)_{\theta,q}\\ &= \{f(x), \text{ strongly } (A_0+A_1) \text{ measurable}: \|f(x)\|_{(A_0,A_1)_{\theta,q}}<\infty\}.\end{aligned}$$

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From this it follows that in fact $(A_0, A_1)_{\theta,q}$ is the only choice for A open to us (unless we envisage an A depending on the structure of the underlying measure space).

In effect we are asking whether there exists a representation in the style of the Lions-Peetre formula:

$$(1)\quad \begin{aligned}&(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}\\ &\qquad = \{f(x), \text{ strongly } (A_0 + A_1)\text{-measurable}: \|f(x)\|_{(A_0,A_1)_{\theta,q}} \in S\}\end{aligned}$$

where S is some class of real valued functions on X, depending on some or maybe all of $p_0, p_1, \theta, q, X, \Sigma, \mu, A_0, A_1$, and $\mathscr{A}$. We note in passing that for $S=L(p(\theta), q)$ the formula (1) is true in each of the three special cases, $q=p(\theta)$, $A_0=A_1$ (provided $p_0 \neq p_1$) and X=single atom. Nevertheless, as we shall show here, (1) is false for at least one choice of $(A_0, A_1, \mathscr{A})$ and (X, Σ, μ) and for "nearly" every cḩoice of parameters, p_0, p_1, θ, q.

THEOREM. *For each choice of parameters $\theta \in (0, 1)$, $p_0, p_1, q \in [1, \infty]$ with $p_0 \neq p_1$ and $q \neq p(\theta)$ there exist $(A_0, A_1, \mathscr{A})$ and (X, Σ, μ) and two strongly (A_0+A_1) measurable functions f and g on X such that,*

(i) $\quad f \in (L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q}, \qquad g \notin (L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q},$

(ii) $\quad \|f(x)\|_{(A_0,A_1)_{\theta,q}} = \|g(x)\|_{(A_0,A_1)_{\theta,q}}$ *for all* $x \in X$.

PROOF. We take $A_0=L^{p_0}_X$, $A_1=L^{p_1}_X$ and $X=R_+$. Since $p_0 \neq p_1$, $(A_0 + A_1)_{\theta,q}=L_X(p(\theta), q)$ and

$$(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,q} = (L^{p_0}_{X^2}, L^{p_1}_{X^2})_{\theta,q} = L_{X^2}(p(\theta), q).$$

Let $L_X(p, q)(L_X(p, q))$ be the space of functions $f(x, y)$, measurable on X^2 such that $\|f(x, \cdot)\|_{L_X(p,q)} \in L_X(p, q)$ as a function of x.

LEMMA. *If $p \neq q$,*

$$L_{R^2_+}(p, q) \neq L_{R_+}(p, q)(L_{R_+}(p, q)),$$

and neither space contains the other.

The Lemma is proved in a later section. Take $p=p(\theta)$. Let $g(x, y)$ be a function on X^2 which belongs to $L_X(p, q)(L_X(p, q))$ but not to $L_{X^2}(p, q)$. As a function of x, $G(x)=\|g(x, \cdot)\|_{L_X(p,q)}$ must belong to $L_X(p, q)$. Let E be a subset of X such that $\|\chi_E\|_{L_X(p,q)}=1$ and let $f(x, y)=G(x)\chi_E(y)$. Then we have that

$$f(x, y) \in L_{X^2}(p, q) \quad \text{and} \quad g(x, y) \notin L_{X^2}(p, q),$$

but also that

$$\|f(x,\cdot)\|_{L_X(p,q)} = \|g(x,\cdot)\|_{L_X(p,q)} \quad \text{for all } x.$$

This completes the proof.

REMARK. It follows immediately that a formula of form (1) can never be generally true if $q \neq p(\theta)$ when p_0 and p_1 are unequal. However one might still hope that in the case $p_0 = p_1$, where the above set of counter-examples is inapplicable, there exists a generalization of the Lions-Peetre formula of form (1) for $q \neq p(\theta)$. Again we show that this is not so. Put $p = p_0 = p_1 = p(\theta)$. Then if $A_0 = A_1$, (1) is true with $S = L_X^p$ for all values of θ and q. So unless we were to admit an S which is allowed to vary its form depending on whether A_0 and A_1 are equal or not, the only possible version of (1) would be with $S = L_X^p$.

In fact if $q \leqq p$,

$$(L^p(A_0), L^p(A_1))_{\theta,q} \subset L^p((A_0, A_1)_{\theta,q})$$

and the reverse inclusion holds for $q \geqq p$. These are readily proved using the integral version of Minkowski's inequality, and the observation that $f(x) \in (L^p(A_0), L^p(A_1))_{\theta,q}$ if and only if

$$\int_0^\infty t^{-\theta q}\left(\int_X K(t,f(x))^p \, d\mu\right)^{q/p} \frac{dt}{t} < \infty.$$

[Here we adopt the standard notation

$$K(t,f(x)) = K(t,f(x), A_0, A_1) = \inf_{g+h=f(x)} (\|g\|_{A_0} + t\,\|h\|_{A_1})$$

for each x.]

To show that these inclusions are not, in general, equalities consider for example the case $q = \infty$. Here

$$(2) \qquad L^p((A_0, A_1)_{\theta,\infty}) \subset (L^p(A_0), L^p(A_1))_{\theta,\infty}.$$

Let $X = R_+$, and $A_0 = L^1_{R_+}$, $A_1 = L^\infty_{R_+}$. Then $K(t,f) = \int_0^t f^*(s)\,ds$ (see [**1**, p. 184]). Let $f(x,\cdot) = x^{-(2-\theta)/p}\chi_{(0,x^{1/p}]}(\cdot)$. Then

$$\begin{aligned} K(t,f(x)) &= tx^{-(2-\theta)/p} \quad \text{for } 0 < t \leqq x^{1/p}, \\ &= x^{-(1-\theta)/p} \quad \text{for } x^{1/p} \leqq t < \infty. \end{aligned}$$

So $\|f\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{t>0} t^{-\theta}K(t,f) = x^{-1/p} \notin L^p$.

However

$$\begin{aligned} \int_X K(t,f(x))^p \, d\mu &= \int_0^{t^p} x^{-(1-\theta)}\,dx + t^p \int_{t^p}^\infty x^{-(2-\theta)}\,dx \\ &= (1/\theta + 1/(1-\theta))t^{\theta p}, \end{aligned}$$

so

$$\sup_{t>0} t^{-\theta}\left(\int_X K(t, f(x))^p\, d\mu\right)^{1/p} < \infty \quad \text{and} \quad f \in (L^p(A_0), L^p(A_1))_{\theta,\infty},$$

proving that (2) is strict. We could equally well have used

$$g(x, \cdot) = e^{-x(1-\theta)}\chi_{(0,e^x]}(\cdot)$$

on $(-\infty, \infty)$ instead of $f(x, \cdot)$ on R_+. $\|g(x, \cdot)\|_{(A_0,A_1)_{\theta,\infty}}=1$ for all x, and as we have already noted $\|f(x, \cdot)\|_{(A_0,A_1)_{\theta,\infty}}=x^{-1/p}$. Thus we may also remark that

$$(L^p(A_0), L^p(A_1))_{\theta,\infty} \not\subset L(q, r)((A_0, A_1)_{\theta,\infty})$$

for all q and r.

Proof of the Lemma. We show that neither of the spaces $A=L_{R_+^2}(p, q)$ and $B=L_{R_+}(p, q)(L_{R_+}(p, q))$ contains the other if $p\neq q$. It is convenient to use the natural quasinorms for the spaces A and B (see [2], [4]),

$$\|f\|_A^* = \|f\|_{p,q}^* = \left(\int_0^\infty t^{q/p-1}f^*(t)^q\, dt\right)^{1/q}, \qquad q < \infty,$$

$$= \sup_{t>0} t^{1/p}f^*(t), \qquad q = \infty,$$

where $f^*(t)$ is the nonincreasing rearrangement on R_+ of the function $f(x, y)$ on R_+^2.

$$\|f\|_B^* = \|(\|f(x, \cdot)\|_{p,q}^*)\|_{p,q}^*,$$

where here $\|\cdot\|_{p,q}^*$ denotes the usual $L_{R_+}(p, q)$ quasinorm. It is first evaluated for $f(x, y)$ as a function of y for each x, and subsequently for the resulting function of x.

(i) Consider $q=\infty$. $g(x, y)=1/(xy)^{1/p}$ is in B but not A. For the reverse noninclusion, define

$$f(x, y) = 0 \qquad \text{for } y > 2x^{-3}\exp(-1/x^2),$$

$$= \exp(1/px^2) \quad \text{for } 0 \leqq y \leqq 2x^{-3}\exp(-1/x^2).$$

$$\|f\|_A^* = \sup_{t>0} t^{1/p}f^*(t)$$

$$= \sup_{s>0}\left(\int_0^s 2x^{-3}\exp(-1/x^2)\, dx\right)^{1/p}\exp(1/ps^2)$$

$$= \sup_{s>0}(\exp(-1/s^2))^{1/p}\exp(1/ps^2) = 1.$$

But

$$\|f(x, \cdot)\|_{p,\infty}^* = [2x^{-3}\exp(-1/x^2)]^{1/p}\exp(1/px^2)$$

$$= [2x^{-3}]^{1/p}.$$

Therefore

$$\|f\|_B^* = \sup_{x>0} x^{1/p}[2x^{-3}]^{1/p} = \infty.$$

(ii) $q<\infty$. Each counterexample function will be constructed similarly to f above. Take a nonnegative function $e(x)$ and a nonnegative nonincreasing continuous function $F(x)$ and define

$$\begin{aligned} f(x, y) &= 0 \qquad \text{for } y > e(x), \\ &= F(x) \quad \text{for } 0 \leqq y \leqq e(x). \end{aligned}$$

Then, if $E(x)=\int_0^x e(t)\,dt$,

$$\begin{aligned} \|f\|_A^* &= \left(\int_0^\infty t^{q/p-1} f^*(t)^q \,dt\right)^{1/q} \\ &= \left(\int_0^\infty E(x)^{q/p-1} F(x)^q e(x)\,dx\right)^{1/q} \end{aligned}$$

and

$$\begin{aligned} \|f(x,\cdot)\|_{p,q}^* &= F(x)\left(\int_0^{e(x)} t^{q/p-1}\,dt\right)^{1/q} \\ &= (p/q)^{1/q} e(x)^{1/p} F(x). \end{aligned}$$

In each of the following cases e and F will be chosen to ensure also that $e(x)^{1/p}F(x)$ is nonincreasing and continuous. Then

$$\|f\|_B^* = (p/q)^{1/q}\left(\int_0^\infty x^{q/p-1} F(x)^q e(x)^{q/p}\,dx\right)^{1/q}.$$

First suppose $p<q$. Put

$$e(x) = \exp x \cdot \exp(\exp x)$$

so

$$E(x) = \exp(\exp x) - \exp 1.$$

Put

$$F(x) = 1/e(x)^{1/p}.$$

$$\begin{aligned} \|f\|_A^{*q} &= \int_0^\infty [E(x)/e(x)]^{q/p-1}\,dx \\ &\leqq \int_0^\infty [\exp(-x)]^{q/p-1}\,dx < \infty. \end{aligned}$$

But $\|f\|_B^{*q}=(p/q)\int_0^\infty x^{q/p-1}\,dx=\infty$. Thus $A\not\subset B$.

Next put $e(x)=1/(x+1)$, so

$$E(x) = \log(x+1) \quad \text{and} \quad F(x) = \min\{1, [\log(x+1)]^{-1/p}\}.$$

$$\|f\|_A^{*q} \geqq \int_{e-1}^\infty [\log(x+1)]^{q/p-1-q/p}(x+1)^{-1}\,dx = \infty.$$

But

$$\|f\|_B^{*q} \sim (p/q)\int_{e-1}^{\infty} x^{q/p-1}[\log(x+1)]^{-q/p}(x+1)^{-q/p}\,dx < \infty$$

proving $B \not\subset A$.

It remains to consider the case $p>q$. Again take $e(x)=1/(x+1)$ so $E(x)=\log(x+1)$. Let

$$\begin{aligned} F(x) &= [\log(x+1)]^{-1/p+\varepsilon/q} \quad \text{for } 0 \leqq x \leqq e-1, \\ &= [\log(x+1)]^{-1/p-\varepsilon/q} \quad \text{for } x \geqq e-1, \end{aligned}$$

with $0<\varepsilon<\min[q/p, 1-q/p]$.

$$\begin{aligned} \|f\|_A^{*q} &= \int_0^{e-1} [\log(x+1)]^{q/p-1}[\log(x+1)]^{-q/p+\varepsilon}(x+1)^{-1}\,dx \\ &\quad + \int_{e-1}^{\infty} [\log(x+1)]^{q/p-1}[\log(x+1)]^{-q/p-\varepsilon}(x+1)^{-1}\,dx \\ &= \int_0^1 r^{-1+\varepsilon}\,dr + \int_1^{\infty} r^{-1-\varepsilon}\,dr < \infty. \end{aligned}$$

$$\begin{aligned} \|f\|_B^{*q} &\geqq (p/q)\int_{e-1}^{\infty} (x+1)^{q/p-1}[\log(x+1)]^{-q/p-\varepsilon}(x+1)^{-q/p}\,dx \\ &= (p/q)\int_1^{\infty} r^{-q/p-\varepsilon}\,dr = \infty. \end{aligned}$$

Therefore $A \not\subset B$.

Finally put $e(x)=\exp x$, so $E(x)=\exp x-1$. Let

$$\begin{aligned} F(x) &= \exp(-x/p) \quad \text{on } [0,1], \\ &= \exp(-x/p)x^{-1/p-\varepsilon/q} \quad \text{on } [1,\infty), \end{aligned}$$

where $0<\varepsilon<1-q/p$. Note that, as always $e(x)^{1/p}F(x)$ is nonincreasing.

$$\begin{aligned} \|f\|_A^{*q} &\geqq \int_1^{\infty} (\exp x - 1)^{q/p-1}x^{-q/p-\varepsilon}(\exp x)^{-q/p+1}\,dx \\ &\geqq \int_1^{\infty} x^{-q/p-\varepsilon}\,dx = \infty. \end{aligned}$$

But

$$\begin{aligned} \|f\|_B^{*q} &= (p/q)\int_0^1 x^{q/p-1}\exp(-qx/p)\exp(qx/p)\,dx \\ &\quad + (p/q)\int_1^{\infty} x^{q/p-1}x^{-q/p-\varepsilon}\,dx \\ &< \infty, \end{aligned}$$

showing that $B \not\subset A$ and completing the proof of the Lemma.

REMARK. In fact the Lions-Peetre formula for $(L^{p_0}(A_0), L^{p_1}(A_1))_{\theta,p(\theta)}$ is true for p_0 and p_1 in the extended range $(0, \infty]$ (see [**4**]). Similarly the Theorem and Lemma presented above remain valid for $p_0, p_1, q \in (0, \infty]$.

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DEPARTMENT OF THEORETICAL MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT, ISRAEL

Current address: Mathématique (Bât. 425), Université de Paris-Sud, Orsay 91405, France