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FIXED POINT THEOREMS IN UNIFORMLY CONVEX BANACH SPACES

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ABSTRACT. The notion of an asymptotic center is used to prove a number of results concerning the existence of fixed points under certain selfmappings of a closed and bounded convex subset of a uniformly convex Banach space.

1. **Introduction.** In this paper we shall assume that X is a Banach space with positive modulus of convexity $\delta(\varepsilon)$ (i.e. X is uniformly convex), where

$$\delta(\varepsilon) = \inf\{1 - \tfrac{1}{2}\|x + y\| : \|x\| \leqq 1, \|y\| \leqq 1, \|x - y\| \geqq \varepsilon\} \quad (0 < \varepsilon \leqq 2).$$

Let $\{u_n : n=1, 2, \cdots\}$ be a bounded sequence in a closed convex subset C of X. As in [**2**] we define

$$r_m(x) = \sup\{\|u_n - x\| : n \geqq m\} \tag{1}$$

and denote by c_m the unique point in C with the property that

$$r_m(c_m) = \inf\{r_m(x) : x \in C\}. \tag{2}$$

It was shown in [**2**] that a point c, called the asymptotic center of $\{u_n\}$ with respect to C, exists such that $c_m \to c$ as $m \to \infty$.

Some basic properties of the asymptotic center are collected in §2 of the present paper. These are then used to obtain a much stronger version of a fixed point theorem proved first in [**2**]. Next, a fixed point theorem for a countable family of commuting mappings, more general than nonexpansive ones, is proved. A special feature of our results is that, given an orbit of an arbitrary point in C, the location of the fixed point is known a priori (as it coincides with the asymptotic center of that orbit) and it has certain uniqueness properties.

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2. **Preliminaries.**

2.1. We shall make use of the mapping r of X into the reals defined by

$$r(x) = \inf\{r_m(x): m = 1, 2, \cdots\}. \tag{3}$$

The mapping r as well as the r_m ($m=1, 2, \cdots$) are continuous on all of X. This follows easily from the fact that $r_m(x) \to r(x)$ as $m \to \infty$ and

$$|r_m(x) - r_m(x')| \leqq \|x - x'\| \qquad (x, x' \in X: m = 1, 2, \cdots). \tag{4}$$

Also, $r_m(y_m) \to r(y)$ as $y_m \to y$; for

$$\begin{aligned}|r_m(y_m) - r(y)| &\leqq |(r_m(y_m) - r_m(y)) + (r_m(y) - r(y))| \\ &\leqq \|y_m - y\| + |r_m(y) - r(y)|.\end{aligned}$$

2.2. For the asymptotic center c of $\{u_n\}$ with respect to C we have

$$x \in C \sim \{c\} \Rightarrow r((x + c)/2) < r(x). \tag{5}$$

Indeed, since $c_n \to c$ [**2**, Theorem 1] there is an N such that, for $n \geqq N$, $\|c_n - x\| > \frac{1}{2}\|c - x\|$. By uniform convexity and the definition of r_n and c_n we thus have for $k \geqq n \geqq N$

$$\|u_k - (x + c_n)/2\| \leqq r_n(x)(1 - \delta(\|x - c\|/2\rho))$$

where ρ is a positive constant (e.g. $\rho = r_1(x) + 1$). Thus

$$r_n((x + c_n)/2) \leqq r_n(x)(1 - \delta(\|x - c\|/2\rho))$$

and therefore

$$r((x + c)/2) \leqq r(x)(1 - \delta(\|x - c\|/2\rho)) < r(x)$$

as claimed.

2.3. From (5) we conclude easily that

$$x \in C \sim \{c\} \Rightarrow r(c) < r(x). \tag{6}$$

For, clearly, $r_m(c_m) \leqq r_m(y)$. Thus $r(c) \leqq r(y)$ for all $y \in C$. If, however, contrary to (6), $r(c) = r(x)$ then with $y = (x+c)/2$ we would have, by (5), $r(c) > r(y)$.

2.4. If for some n_0, $n \geqq n_0 \Rightarrow \|u_n - z\| \leqq \|u_n - c\|$ then $z = c$. Indeed if $m \geqq n_0$ the above implies that $r_m(z) \leqq r_m(c)$ and, therefore, $r(z) \leqq r(c)$ which, by (6) is only possible if $z = c$.

2.5. In the special case when X is a Hilbert space then $c \in \operatorname{cl} \operatorname{co}\{u_n\}$, the closed convex hull of $\{u_n\}$.

PROOF. If $v \in X$ is not in $\operatorname{cl} \operatorname{co}\{u_n\}$ then there is a nearest point c' to v in $\operatorname{cl} \operatorname{co}\{u_n\}$. The hyperplane $\{x \in X: \langle x - c', c' - v\rangle = 0\}$ separates v from $\operatorname{cl} \operatorname{co}\{u_n\}$ and we may clearly assume that $\operatorname{Re}\langle u_n - c', c' - v\rangle > 0$

$(n=1, 2, \cdots)$. Hence

$$\|u_n - v\|^2 = \|u_n - c'\|^2 + \|c' - v\|^2 + 2\,\mathrm{Re}\langle u_n - c', c' - v\rangle$$
$$> \|u_n - c'\|^2 \qquad (n = 1, 2, \cdots).$$

It follows from 2.4 above that v cannot coincide with the asymptotic center c.

3. **Fixed point theorems.** A well-known theorem due to Browder [**1**], Göhde [**3**] and, in a somewhat stronger form, to Kirk [**4**] states that every nonexpansive mapping of a closed and bounded convex set, in a uniformly convex Banach space, into itself has a fixed point. The next theorem, while assuming substantially less on the mapping, proves the existence of a fixed point having a preassigned location.

THEOREM 1. *Let $f: C\to C$ be a mapping of a closed convex subset C of a uniformly convex Banach space into itself and let $\{f^n(x): n=1, 2, \cdots\}$ be a bounded sequence of iterates of some $x \in C$ having the asymptotic center c with respect to C. If an N exists such that*

$$\|f^n(x) - f(c)\| \leqq \|f^{n-1}(x) - c\| \qquad (n > N) \tag{7}$$

then $f(c)=c$.

PROOF. Let $c'=f(c)$. Then

$$\|f^n(x) - c'\| = \|f^n(x) - f(c)\| \leqq \|f^{n-1}(x) - c\|, \qquad n > N.$$

Hence $r_n(c')\leqq r_{n-1}(c)$ $(n>N)$ and, therefore, $r(c')\leqq r(c)$. By (6) this can only happen when $c=c'$, i.e. $f(c)=c$.

3.1. REMARKS. (1) The above theorem is stronger than Theorem 2 of [**2**] in that condition (7) requires less than its counterpart there which requires that the same inequality be satisfied not only for c but for all points v of some neighborhood V of c.

(2) A mapping $f: C\to C$ has a fixed point if the following condition is satisfied:

For all $x, y \in C$ there is an $N=N(x, y)$ such that

$$\|f^n(x) - f(y)\| \leqq \|f^{n-1}(x) - y\| \qquad (n > N). \tag{8}$$

Indeed, given an arbitrary $x \in C$, choose $y=c$, the asymptotic center of $\{f^n(x)\}$. Then $f(c)=c$.

(Note that in the corresponding statement in [**2**] the misprint occurs involving the replacement of $f(y)$ and y by $f^n(y)$ and $f^{n-1}(y)$ respectively.)

3.2. If (7) is replaced by

$$\|f^n(x) - f^m(c)\| \leqq \|f^{n-1}(x) - c\| \qquad (n > N) \tag{9}$$

where m is a fixed positive integer then the argument used in the proof of Theorem 1 yields the fact that $f^m(c)=c$ i.e. c is a periodic point. The proof of the following theorem is based in part on this observation.

THEOREM 2. *Let C and f be as in Theorem 1 and suppose that for each $x \in C$ there is an $N=N(x)$ and $m=m(x)$ such that, whenever $c=c(x)$ is the asymptotic center of $\{f^n(x)\}$ and $c \notin \{f^n(x):n>N\}$,*

$$\|f^n(x) - f^m(c)\| \leqq \|f^{n-1}(x) - c\| \qquad (n > N) \tag{10}$$

with strict inequality in the case when $f^{n-1}(x) \neq c$.

Then, for each $x \in C, f(c)=c\ (=c(x))$.

PROOF. As observed before, if $x \in C$ and $c=c(x)$, $f^m(c)=c$. Suppose $y=c\neq f(c)$ so that $m>1$. Let $N'=N(y)$, $m'=m(y)$ and $c'=c(y)$. Since, again $f^{m'}(c')=c'$ we obtain from (10) $\|f^n(y)-c'\|<\|f^{n-1}-c'\|$ $(n>N')$ showing that $\|f^{N'+k}(y)-c'\|$ decreases with increasing k. On the other hand

$$\|f^{N'}(y) - c'\| = \|f^{N'+m}(y) - c'\| < \|f^{N'+m-1}(y) - c'\|$$

which is impossible. Thus $c=f(c)$ proving the theorem.

3.3. In the case of nonexpansive mappings the asymptotic center c of $\{f^n(x)\}$ for an arbitrary x in C is next shown to be the fixed point of f which is closest to $\{f^n(x):n=0, 1, \cdots\}$.

PROPOSITION 2. *Let $f: C\rightarrow C$ be a nonexpansive mapping of the closed and bounded convex set C into itself. If $x \in C$ and c is the asymptotic center of $\{f^n(x):n=0, 1, \cdots\}$ then*

$$r(c) = \inf\{\|f^n(x) - c\|:n = 0, 1, \cdots\} \leqq \inf\{\|f^n(x) - \xi\|:n = 0, 1, \cdots\}$$

for each fixed point ξ of f.

PROOF. From $f(\xi)=\xi$ it follows that $\|f^{n+1}(x)-\xi\|\leqq\|f^n(x)-\xi\|$ implying that $r_n(\xi)=\|f^n(x)-\xi\|$ and $r(\xi)=\inf\{\|f^n(x)-\xi\|:n=0, 1, \cdots\}$. The conclusion now follows directly from (6).

4. **Fixed points common to certain families of mappings.** In this section we first prove the existence of a common fixed point to a sequence of commuting mappings satisfying conditions considerably weaker than nonexpansiveness. This is accomplished by producing a sequence of asymptotic centers $\{c^{(m)}\}$ and showing that its asymptotic center has the desired property. In Hilbert space such a fixed point has a nearest point property analogous to that stated in Proposition 1 for a single mapping. Finally Browder's theorem **[1]** on the existence of a common fixed point for a family of commuting nonexpansive mappings is shown to hold for the wider class satisfying condition (8).

THEOREM 3. *Let C be a closed and bounded convex set in a uniformly convex Banach space and $\{f_n : n=0, 1, \cdots\}$ a sequence of commuting mappings of C into itself. Let $c^{(1)}$ be the asymptotic center of $\{f_0^n(x)\}$ for some $x=c^{(0)} \in C$ and, recursively, let $c^{(n)}$ be the asymptotic center of $\{f_{n-1}^m(c^{(n-1)})\}$. Let c be the asymptotic center of $\{c^{(n)} : n=0, 1, \cdots\}$ and suppose that*

$$\|f_k(c^{(l)}) - f_k(c)\| \leqq \|c^{(l)} - c\| \qquad (l \geqq l_k > k;\ k = 0, 1, \cdots) \tag{11}$$

$$\|f_k^n(c^{(l)}) - f_k(c^{(l+1)})\| \leqq \|f_k^{n-1}(c^{(l)}) - c^{(l+1)}\| \qquad (n \geqq N(l, k);\ l, k = 0, 1, \cdots) \tag{12}$$

and

$$\|f_k f_{l-1}^n(c^{(l-1)}) - f_k(c^{(l)})\| \leqq \|f_{l-1}^n(c^{(l-1)}) - c^{(l)}\| \qquad (n \geqq N(l, k);\ l, k = 1, 2, \cdots). \tag{13}$$

Then $f_k(c)=c$ for all $k=0, 1, \cdots$.

PROOF. It suffices to show that for all $l>k$, $k=0, 1, \cdots$,

$$f_k(c^{(l)}) = c^{(l)}. \tag{14}$$

Indeed, if this is true then by (11)

$$\|c^{(l)} - f_k(c)\| = \|f_k(c^{(l)}) - f_k(c)\| \leqq \|c^{(l)} - c\|$$

for $l \geqq l_k > k$ and $k=0, 1, \cdots$. By 2.4, then, $f_k(c)=c$. To prove (14) note that $f_k(c^{(k+1)})=c^{(k+1)}$ by Theorem 1 as a consequence of (12). Assuming then that $f_k(c^{(l-1)})=c^{(l-1)}$ for $l-1>k$ we get from (13)

$$\begin{aligned}\|f_{l-1}^n(c^{(l-1)}) - f_k(c^{(l)})\| &= \|f_k f_{l-1}^n(c^{(l-1)}) - f_k(c^{(l)})\| \\ &\leqq \|f_{l-1}^n(c^{(l-1)}) - c^{(l)}\| \qquad (n \geqq N(l, k))\end{aligned}$$

implying, as before, that $f_k(c^{(l)})=c^{(l)}$.

PROPOSITION 3. *Let X be a Hilbert space and $\{f_n : n=0, 1, \cdots\}$ a sequence of commuting nonexpansive mappings of a closed and bounded convex subset $C \subset X$ into itself. Let $\{c^{(n)}\}$ and c be as in Theorem 3. If ξ is any common fixed point of the f_n's then*

$$\inf\{\|c^{(n)} - c\| : n = 0, 1, \cdots\} \leqq \inf\{\|c^{(n)} - \xi\| : n = 0, 1, \cdots\}.$$

PROOF. It follows from 2.5 that $\|c^{(n+1)} - \xi\| \leqq \|c^{(n)} - \xi\|$, $n=0, 1, \cdots$. For

$$\|f_n^k(c^{(n)}) - f_n^k(\xi)\| \leqq \|c^{(n)} - \xi\| \qquad (n = 0, 1, \cdots;\ k = 1, 2, \cdots)$$

and, therefore $c^{(n+1)}$, being in cl co$\{f_n^k(c^{(n)}) : k=1, 2, \cdots\}$, is in the closed ball $B(\xi, \|c^{(n)}-\xi\|)$. Hence the conclusion of the proposition follows in the same manner as that of Proposition 2.

THEOREM 4. *Let* $\{f_a : a \in A\}$ *be a family of commuting continuous mappings of a closed and bounded convex set* C *(in a uniformly convex Banach space* X*) into itself and suppose that for each* $a \in A$ *and* $x, y \in C$ *there is an* $N(x, y; a)$ *such that for* $n > N(x, y; a)$

$$\|f_a^{n+1}(x) - f_a(y)\| \leqq \|f_a^n(x) - y\|. \tag{15}$$

Then there is a $\xi \in C$ *such that* $f_a(\xi) = \xi$ $(a \in A)$.

PROOF. Let F_a be the set of fixed points of f_a. We then have to show that $\bigcap \{F_a : a \in A\} \neq \varnothing$. A standard, and often used, argument shows that the commuting property implies that the F_a have the finite intersection property. Thus it suffices to show that each F_a is weakly compact. In view of the reflexivity of X it suffices then to show that each F_a is a nonempty closed and convex subset of C. From 3.1(2) we know that $F_a \neq \varnothing$ $(a \in A)$. Let then $y, z \in F_a$ and $u = \lambda y + (1-\lambda)z$ with $0 < \lambda < 1$. Choose $n > \max(N(y, u; a), N(z, u; a))$. Then

$$\|y - f_a(u)\| = \|f_a^{n+1}(y) - f_a(u)\| \leqq \|f_a^n(y) - u\| = \|y - u\|$$

and, similarly, $\|z - f_a(u)\| \leqq \|z - u\|$. This, however, in any strictly convex Banach space is only possible when $f_a(u) = u$ i.e. $u \in F_a$. Closedness of each F_a is obvious.

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