

## FIXED POINT THEOREMS IN UNIFORMLY CONVEX BANACH SPACES

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**ABSTRACT.** The notion of an asymptotic center is used to prove a number of results concerning the existence of fixed points under certain selfmappings of a closed and bounded convex subset of a uniformly convex Banach space.

**1. Introduction.** In this paper we shall assume that  $X$  is a Banach space with positive modulus of convexity  $\delta(\varepsilon)$  (i.e.  $X$  is uniformly convex), where

$$\delta(\varepsilon) = \inf\{1 - \frac{1}{2} \|x + y\| : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}$$

$(0 < \varepsilon \leq 2).$

Let  $\{u_n : n=1, 2, \dots\}$  be a bounded sequence in a closed convex subset  $C$  of  $X$ . As in [2] we define

$$(1) \quad r_m(x) = \sup\{\|u_n - x\| : n \geq m\}$$

and denote by  $c_m$  the unique point in  $C$  with the property that

$$(2) \quad r_m(c_m) = \inf\{r_m(x) : x \in C\}.$$

It was shown in [2] that a point  $c$ , called the asymptotic center of  $\{u_n\}$  with respect to  $C$ , exists such that  $c_m \rightarrow c$  as  $m \rightarrow \infty$ .

Some basic properties of the asymptotic center are collected in §2 of the present paper. These are then used to obtain a much stronger version of a fixed point theorem proved first in [2]. Next, a fixed point theorem for a countable family of commuting mappings, more general than nonexpansive ones, is proved. A special feature of our results is that, given an orbit of an arbitrary point in  $C$ , the location of the fixed point is known a priori (as it coincides with the asymptotic center of that orbit) and it has certain uniqueness properties.

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## 2. Preliminaries.

2.1. We shall make use of the mapping  $r$  of  $X$  into the reals defined by

$$(3) \quad r(x) = \inf\{r_m(x) : m = 1, 2, \dots\}.$$

The mapping  $r$  as well as the  $r_m$  ( $m=1, 2, \dots$ ) are continuous on all of  $X$ . This follows easily from the fact that  $r_m(x) \rightarrow r(x)$  as  $m \rightarrow \infty$  and

$$(4) \quad |r_m(x) - r_m(x')| \leq \|x - x'\| \quad (x, x' \in X : m = 1, 2, \dots).$$

Also,  $r_m(y_m) \rightarrow r(y)$  as  $y_m \rightarrow y$ ; for

$$\begin{aligned} |r_m(y_m) - r(y)| &\leq |(r_m(y_m) - r_m(y)) + (r_m(y) - r(y))| \\ &\leq \|y_m - y\| + |r_m(y) - r(y)|. \end{aligned}$$

2.2. For the asymptotic center  $c$  of  $\{u_n\}$  with respect to  $C$  we have

$$(5) \quad x \in C \sim \{c\} \Rightarrow r((x+c)/2) < r(x).$$

Indeed, since  $c_n \rightarrow c$  [2, Theorem 1] there is an  $N$  such that, for  $n \geq N$ ,  $\|c_n - x\| > \frac{1}{2}\|c - x\|$ . By uniform convexity and the definition of  $r_n$  and  $c_n$  we thus have for  $k \geq n \geq N$

$$\|u_k - (x + c_n)/2\| \leq r_n(x)(1 - \delta(\|x - c\|/2\rho))$$

where  $\rho$  is a positive constant (e.g.  $\rho = r_1(x) + 1$ ). Thus

$$r_n((x + c_n)/2) \leq r_n(x)(1 - \delta(\|x - c\|/2\rho))$$

and therefore

$$r((x + c)/2) \leq r(x)(1 - \delta(\|x - c\|/2\rho)) < r(x)$$

as claimed.

2.3. From (5) we conclude easily that

$$(6) \quad x \in C \sim \{c\} \Rightarrow r(c) < r(x).$$

For, clearly,  $r_m(c_m) \leq r_m(y)$ . Thus  $r(c) \leq r(y)$  for all  $y \in C$ . If, however, contrary to (6),  $r(c) = r(x)$  then with  $y = (x+c)/2$  we would have, by (5),  $r(c) > r(y)$ .

2.4. If for some  $n_0$ ,  $n \geq n_0 \Rightarrow \|u_n - z\| \leq \|u_n - c\|$  then  $z = c$ . Indeed if  $m \geq n_0$  the above implies that  $r_m(z) \leq r_m(c)$  and, therefore,  $r(z) \leq r(c)$  which, by (6) is only possible if  $z = c$ .

2.5. In the special case when  $X$  is a Hilbert space then  $c \in \text{cl co}\{u_n\}$ , the closed convex hull of  $\{u_n\}$ .

PROOF. If  $v \in X$  is not in  $\text{cl co}\{u_n\}$  then there is a nearest point  $c'$  to  $v$  in  $\text{cl co}\{u_n\}$ . The hyperplane  $\{x \in X : \langle x - c', c' - v \rangle = 0\}$  separates  $v$  from  $\text{cl co}\{u_n\}$  and we may clearly assume that  $\text{Re}\langle u_n - c', c' - v \rangle > 0$

( $n=1, 2, \dots$ ). Hence

$$\begin{aligned} \|u_n - v\|^2 &= \|u_n - c'\|^2 + \|c' - v\|^2 + 2 \operatorname{Re}\langle u_n - c', c' - v \rangle \\ &> \|u_n - c'\|^2 \quad (n = 1, 2, \dots). \end{aligned}$$

It follows from 2.4 above that  $v$  cannot coincide with the asymptotic center  $c$ .

**3. Fixed point theorems.** A well-known theorem due to Browder [1], Göhde [3] and, in a somewhat stronger form, to Kirk [4] states that every nonexpansive mapping of a closed and bounded convex set, in a uniformly convex Banach space, into itself has a fixed point. The next theorem, while assuming substantially less on the mapping, proves the existence of a fixed point having a preassigned location.

**THEOREM 1.** *Let  $f: C \rightarrow C$  be a mapping of a closed convex subset  $C$  of a uniformly convex Banach space into itself and let  $\{f^n(x): n=1, 2, \dots\}$  be a bounded sequence of iterates of some  $x \in C$  having the asymptotic center  $c$  with respect to  $C$ . If an  $N$  exists such that*

$$(7) \quad \|f^n(x) - f(c)\| \leq \|f^{n-1}(x) - c\| \quad (n > N)$$

then  $f(c) = c$ .

**PROOF.** Let  $c' = f(c)$ . Then

$$\|f^n(x) - c'\| = \|f^n(x) - f(c)\| \leq \|f^{n-1}(x) - c\|, \quad n > N.$$

Hence  $r_n(c') \leq r_{n-1}(c)$  ( $n > N$ ) and, therefore,  $r(c') \leq r(c)$ . By (6) this can only happen when  $c = c'$ , i.e.  $f(c) = c$ .

**3.1. REMARKS.** (1) The above theorem is stronger than Theorem 2 of [2] in that condition (7) requires less than its counterpart there which requires that the same inequality be satisfied not only for  $c$  but for all points  $v$  of some neighborhood  $V$  of  $c$ .

(2) A mapping  $f: C \rightarrow C$  has a fixed point if the following condition is satisfied:

For all  $x, y \in C$  there is an  $N = N(x, y)$  such that

$$(8) \quad \|f^n(x) - f(y)\| \leq \|f^{n-1}(x) - y\| \quad (n > N).$$

Indeed, given an arbitrary  $x \in C$ , choose  $y = c$ , the asymptotic center of  $\{f^n(x)\}$ . Then  $f(c) = c$ .

(Note that in the corresponding statement in [2] the misprint occurs involving the replacement of  $f(y)$  and  $y$  by  $f^n(y)$  and  $f^{n-1}(y)$  respectively.)

**3.2.** If (7) is replaced by

$$(9) \quad \|f^n(x) - f^m(c)\| \leq \|f^{n-1}(x) - c\| \quad (n > N)$$

where  $m$  is a fixed positive integer then the argument used in the proof of Theorem 1 yields the fact that  $f^m(c)=c$  i.e.  $c$  is a periodic point. The proof of the following theorem is based in part on this observation.

**THEOREM 2.** *Let  $C$  and  $f$  be as in Theorem 1 and suppose that for each  $x \in C$  there is an  $N=N(x)$  and  $m=m(x)$  such that, whenever  $c=c(x)$  is the asymptotic center of  $\{f^n(x)\}$  and  $c \notin \{f^n(x):n>N\}$ ,*

$$(10) \quad \|f^n(x) - f^m(c)\| \leq \|f^{n-1}(x) - c\| \quad (n > N)$$

with strict inequality in the case when  $f^{n-1}(x) \neq c$ .

Then, for each  $x \in C$ ,  $f(c)=c$  ( $=c(x)$ ).

**PROOF.** As observed before, if  $x \in C$  and  $c=c(x)$ ,  $f^m(c)=c$ . Suppose  $y=c \neq f(c)$  so that  $m>1$ . Let  $N'=N(y)$ ,  $m'=m(y)$  and  $c'=c(y)$ . Since, again  $f^{m'}(c')=c'$  we obtain from (10)  $\|f^{N'}(y)-c'\| < \|f^{N'-1}-c'\|$  ( $n>N'$ ) showing that  $\|f^{N'+k}(y)-c'\|$  decreases with increasing  $k$ . On the other hand

$$\|f^{N'}(y) - c'\| = \|f^{N'+m}(y) - c'\| < \|f^{N'+m-1}(y) - c'\|$$

which is impossible. Thus  $c=f(c)$  proving the theorem.

3.3. In the case of nonexpansive mappings the asymptotic center  $c$  of  $\{f^n(x)\}$  for an arbitrary  $x$  in  $C$  is next shown to be the fixed point of  $f$  which is closest to  $\{f^n(x):n=0, 1, \dots\}$ .

**PROPOSITION 2.** *Let  $f:C \rightarrow C$  be a nonexpansive mapping of the closed and bounded convex set  $C$  into itself. If  $x \in C$  and  $c$  is the asymptotic center of  $\{f^n(x):n=0, 1, \dots\}$  then*

$$r(c) = \inf\{\|f^n(x) - c\| : n = 0, 1, \dots\} \leq \inf\{\|f^n(x) - \xi\| : n = 0, 1, \dots\}$$

for each fixed point  $\xi$  of  $f$ .

**PROOF.** From  $f(\xi)=\xi$  it follows that  $\|f^{n+1}(x)-\xi\| \leq \|f^n(x)-\xi\|$  implying that  $r_n(\xi)=\|f^n(x)-\xi\|$  and  $r(\xi)=\inf\{\|f^n(x)-\xi\| : n=0, 1, \dots\}$ . The conclusion now follows directly from (6).

**4. Fixed points common to certain families of mappings.** In this section we first prove the existence of a common fixed point to a sequence of commuting mappings satisfying conditions considerably weaker than nonexpansiveness. This is accomplished by producing a sequence of asymptotic centers  $\{c^{(m)}\}$  and showing that its asymptotic center has the desired property. In Hilbert space such a fixed point has a nearest point property analogous to that stated in Proposition 1 for a single mapping. Finally Browder's theorem [1] on the existence of a common fixed point for a family of commuting nonexpansive mappings is shown to hold for the wider class satisfying condition (8).

**THEOREM 3.** *Let  $C$  be a closed and bounded convex set in a uniformly convex Banach space and  $\{f_n:n=0, 1, \dots\}$  a sequence of commuting mappings of  $C$  into itself. Let  $c^{(1)}$  be the asymptotic center of  $\{f_0^n(x)\}$  for some  $x=c^{(0)} \in C$  and, recursively, let  $c^{(n)}$  be the asymptotic center of  $\{f_{n-1}^m(c^{(n-1)})\}$ . Let  $c$  be the asymptotic center of  $\{c^{(n)}:n=0, 1, \dots\}$  and suppose that*

$$(11) \quad \|f_k(c^{(l)}) - f_k(c)\| \leq \|c^{(l)} - c\| \quad (l \geq l_k > k; k = 0, 1, \dots)$$

$$(12) \quad \|f_k^n(c^{(l)}) - f_k(c^{(l+1)})\| \leq \|f_k^{n-1}(c^{(l)}) - c^{(l+1)}\|$$

$$(n \geq N(l, k); l, k = 0, 1, \dots)$$

and

$$(13) \quad \|f_k f_{l-1}^n(c^{(l-1)}) - f_k(c^{(l)})\| \leq \|f_{l-1}^n(c^{(l-1)}) - c^{(l)}\|$$

$$(n \geq N(l, k); l, k = 1, 2, \dots).$$

Then  $f_k(c) = c$  for all  $k=0, 1, \dots$ .

**PROOF.** It suffices to show that for all  $l > k, k=0, 1, \dots$ ,

$$(14) \quad f_k(c^{(l)}) = c^{(l)}.$$

Indeed, if this is true then by (11)

$$\|c^{(l)} - f_k(c)\| = \|f_k(c^{(l)}) - f_k(c)\| \leq \|c^{(l)} - c\|$$

for  $l \geq l_k > k$  and  $k=0, 1, \dots$ . By 2.4, then,  $f_k(c) = c$ . To prove (14) note that  $f_k(c^{(k+1)}) = c^{(k+1)}$  by Theorem 1 as a consequence of (12). Assuming then that  $f_k(c^{(l-1)}) = c^{(l-1)}$  for  $l-1 > k$  we get from (13)

$$\|f_{l-1}^n(c^{(l-1)}) - f_k(c^{(l)})\| = \|f_k f_{l-1}^n(c^{(l-1)}) - f_k(c^{(l)})\|$$

$$\leq \|f_{l-1}^n(c^{(l-1)}) - c^{(l)}\| \quad (n \geq N(l, k))$$

implying, as before, that  $f_k(c^{(l)}) = c^{(l)}$ .

**PROPOSITION 3.** *Let  $X$  be a Hilbert space and  $\{f_n:n=0, 1, \dots\}$  a sequence of commuting nonexpansive mappings of a closed and bounded convex subset  $C \subset X$  into itself. Let  $\{c^{(n)}\}$  and  $c$  be as in Theorem 3. If  $\xi$  is any common fixed point of the  $f_n$ 's then*

$$\inf\{\|c^{(n)} - c\|:n = 0, 1, \dots\} \leq \inf\{\|c^{(n)} - \xi\|:n = 0, 1, \dots\}.$$

**PROOF.** It follows from 2.5 that  $\|c^{(n+1)} - \xi\| \leq \|c^{(n)} - \xi\|, n=0, 1, \dots$ . For

$$\|f_n^k(c^{(n)}) - f_n^k(\xi)\| \leq \|c^{(n)} - \xi\| \quad (n = 0, 1, \dots; k = 1, 2, \dots)$$

and, therefore  $c^{(n+1)}$ , being in  $\text{cl } \{f_n^k(c^{(n)}):k=1, 2, \dots\}$ , is in the closed

ball  $B(\xi, \|c^{(n)} - \xi\|)$ . Hence the conclusion of the proposition follows in the same manner as that of Proposition 2.

**THEOREM 4.** *Let  $\{f_a : a \in A\}$  be a family of commuting continuous mappings of a closed and bounded convex set  $C$  (in a uniformly convex Banach space  $X$ ) into itself and suppose that for each  $a \in A$  and  $x, y \in C$  there is an  $N(x, y; a)$  such that for  $n > N(x, y; a)$*

$$(15) \quad \|f_a^{n+1}(x) - f_a(y)\| \leq \|f_a^n(x) - y\|.$$

*Then there is a  $\xi \in C$  such that  $f_a(\xi) = \xi$  ( $a \in A$ ).*

**PROOF.** Let  $F_a$  be the set of fixed points of  $f_a$ . We then have to show that  $\bigcap \{F_a : a \in A\} \neq \emptyset$ . A standard, and often used, argument shows that the commuting property implies that the  $F_a$  have the finite intersection property. Thus it suffices to show that each  $F_a$  is weakly compact. In view of the reflexivity of  $X$  it suffices then to show that each  $F_a$  is a nonempty closed and convex subset of  $C$ . From 3.1(2) we know that  $F_a \neq \emptyset$  ( $a \in A$ ). Let then  $y, z \in F_a$  and  $u = \lambda y + (1 - \lambda)z$  with  $0 < \lambda < 1$ . Choose  $n > \max(N(y, u; a), N(z, u; a))$ . Then

$$\|y - f_a(u)\| = \|f_a^{n+1}(y) - f_a(u)\| \leq \|f_a^n(y) - u\| = \|y - u\|$$

and, similarly,  $\|z - f_a(u)\| \leq \|z - u\|$ . This, however, in any strictly convex Banach space is only possible when  $f_a(u) = u$  i.e.  $u \in F_a$ . Closedness of each  $F_a$  is obvious.

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