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INERTIAL h-COBORDISMS WITH FINITE CYCLIC FUNDAMENTAL GROUP

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ABSTRACT. For M a PL n-manifold, $n \geqq 5$, let $I(M)$ be the subset of torsions $\sigma \in \mathrm{Wh}(\pi_1 M)$ such that the h-cobordism W constructed from M with torsion σ has its other boundary component PL homeomorphic to M. We present three techniques dealing with the determination of $I(M)$ and apply them when $\pi_1 M = Z_q$. We prove: (1) If n is even, $\pi_1 M \simeq Z_q$, q odd, then $I(M) = \mathrm{Wh}(\pi_1 M)$. (2) If n is odd, then there exists M with $\pi_1 M \simeq Z_q$ such that $I(M) = \mathrm{Wh}(\pi_1 M)$.

1. **Introduction.** We first establish our notation. For convenience we work in the piecewise linear category; similar results hold in the differential and topological categories. M and N will denote closed manifolds of dimension $n \geqq 5$. We use W (and W') to denote a compact $(n+1)$-manifold which is an h-cobordism between its two boundary components W_0, W_1. "$\simeq$" will denote "is PL homeomorphic to" or "is (group) isomorphic to", depending on the context. We call an h-cobordism W inertial if $W_0 \simeq W_1$. $I(M)$ will denote the subset of $\mathrm{Wh}(\pi_1 M)$ consisting of torsions σ (we measure our torsions in the domain space except where indicated otherwise) such that the h-cobordism $W(M, \sigma)$ with $W_0 = M$ and the torsion of (W, M) (i.e. of $M \subsetneq W$) equal to σ is inertial; i.e. $W_1 \simeq M$. Recall that $W(M, \sigma)$ is determined up to a PL homeomorphism which is the identity on M [**5**].

One of the principal tools of geometric topology is the s-cobordism theorem. Frequently it is used in a context where one needs only to know that a particular h-cobordism is inertial. Thus a determination of $I(M)$ becomes of interest. In this note we present three techniques which are relevant to this problem and apply them to the case where $\pi_1 M \simeq Z_q$, the finite cyclic group of order q. In this case we show that if q is odd and n is even, then $I(M) = \mathrm{Wh}(\pi_1 M)$. If n is odd, the example of lens spaces

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shows that $I(M)$ may be 0 [**5**]. We show that if n odd, q arbitrary, then there is a manifold M with $\pi_1 M \simeq Z_q$ such that $I(M) = \mathrm{Wh}(\pi_1 M)$.

2. **Constructions and applications.** Our first technique involves an application of surgery theory; for further details on surgery, see [**7**]. Our terminology will follow [7]. Let $S_{\mathrm{PL}}(M)$ denote the (simple) homotopy triangulations of M. This has a distinguished element $[1_M] = [f]$, where $f: M' \to M$ is a simple homotopy equivalence that is homotopic to a PL homeomorphism. Let

$$H^m(\pi) = \{\tau \in \mathrm{Wh}(\pi) : \tau = (-1)^m \bar{\tau}\} / \{\tau + (-1)^m \bar{\tau}\},$$

where $\bar{\tau}$ denotes the conjugate of τ. Rothenburg (cf. [7]) has exhibited an exact sequence, three terms of which are

$$H^{n+2}(\pi) \xrightarrow{s} L^s_{n+1}(\pi) \xrightarrow{t} L^h_{n+1}(\pi).$$

Wall [7, p. 108] describes an action of $L^s_{n+1}(\pi_1 M)$ on $S_{\mathrm{PL}}(M)$, which we denote by $\cdot$. In particular, the image of $H^{n+2}(\pi_1 M)$ under s acts on $[1_M] \in S_{\mathrm{PL}}(M)$ as follows. If $\sigma \in \mathrm{Wh}(\pi_1 M)$ satisfies $\sigma = (-1)^n \bar{\sigma}$, then σ represents an element $[\sigma]$ of $H^{n+2}(\pi_1 M)$; $s([\sigma]) \cdot [1_M]$ is represented by $ri: W_1 \to W_0 = M$, where $W = W(M, \sigma)$, $i: W_1 \to W$ is the inclusion and $r: W \to W_0$ is a deformation retraction. The condition $\sigma = (-1)^n \bar{\sigma}$ is required for ri to be a simple homotopy equivalence. If $s([\sigma]) \cdot [1_M] = [1_M]$, then ri is homotopic to a PL homeomorphism and $W_1 \simeq M$. In general, determining this action is a highly nontrivial problem; however, the difficulties collapse under appropriate algebraic assumptions.

Proposition 1. *Let $n \geqq 5$ be even and suppose conjugation is trivial in $\mathrm{Wh}(\pi_1 M)$. Assume also that the map $t: L^s_{n+1}(\pi_1 M) \to L^h_{n+1}(\pi_1 M)$ is injective. Then $I(M) = \mathrm{Wh}(\pi_1 M)$.*

Proof. Since n is even and conjugation is trivial, each $\sigma \in \mathrm{Wh}(\pi_1 M)$ satisfies $\sigma = (-1)^n \bar{\sigma}$. If $s([\sigma]) \cdot [1_M] = [1_M]$, then $\sigma \in I(M)$. But $\mathrm{im}\, s = \ker t = 0$.

Corollary 1. *If $n \geqq 5$ is even and $\pi_1 M \simeq Z_q$, q odd, then $I(M) =$ $\mathrm{Wh}(\pi_1 M)$.*

Proof. Conjugation is trivial in $\mathrm{Wh}(Z_q)$ [**2**]. A. Bak [**1**] has shown that $L^s_{n+1}(\pi) = 0 = L^h_{n+1}(\pi)$ for π a finite abelian group of odd order, hence for $\pi = Z_q$.

Remarks. 1. The ideas behind Proposition 1 have been known for some time. What was needed to apply them were computations of Wall groups such as Bak's.

2. Note that Corollary 1 is nontrivial since $\mathrm{Wh}(\mathbf{Z}_q)$ is a direct sum of $\phi(q)/2-1$ copies of $\mathbf{Z}$, $q\geqq 3$, where ϕ is the Euler ϕ-function [2].

3. $2\mathrm{Wh}(\pi_1 M)=\{\sigma\in\mathrm{Wh}(\pi_1 M):\sigma=2\tau\}$ is easily seen to lie in $I(M)$ via the doubling construction of Milnor [5] when conjugation is trivial. In this case $H^{n+2}(\pi_1 M)=\mathrm{Wh}(\pi_1 M)/2\mathrm{Wh}(\pi_1 M)$.

4. Let $\pi_1 M$ be a finite abelian group of odd order, $C=\{\sigma\in\mathrm{Wh}(\pi_1 M):\sigma=\bar{\sigma}\}$. Then the proof of Proposition 1 shows that $C\subset I(M)$.

When n is odd, the technique above appears to be more difficult to apply; it is useless for $\pi_1 M\simeq\mathbf{Z}_q$. A relevant question for n odd is one of realizability: Given $\sigma\in\mathrm{Wh}(\pi)$, does there exist a manifold M with $\pi_1 M=\pi$ and $\sigma\in I(M)$? In particular, can one find M such that $\mathrm{Wh}(\pi_1 M)=I(M)$? The Milnor doubling construction yields no examples when $\sigma=\bar{\sigma}$, as in $\mathrm{Wh}(\mathbf{Z}_q)$. However, a minor modification of it (which is also related to Farrell's fibering theorem [3]) does lead to some results.

Suppose W is an h-cobordism with $W_0=M$, $W_1=N$ and W' is an h-cobordism with $W'_0=N$ and $W'_1\simeq M$. Suppose $f:N\to N$ is a PL homeomorphism. Let $\sigma,\sigma'\in\mathrm{Wh}(\pi_1 N)$ be the torsions of (W,M), (W',N) as measured in $\mathrm{Wh}(\pi_1 N)$. That is, if σ_1 and σ'_1 denote the torsions of (W,M) and (W',N) measured in $\mathrm{Wh}(\pi_1 W)$ and $\mathrm{Wh}(\pi_1 W')$, respectively, and $r_N:W\to N$, $r'_N:W'\to N$ are deformation retractions, then $(r_N)_\#\sigma_1=\sigma$ and $(r'_N)_\#\sigma'_1=\sigma'$; it turns out to be technically simpler to work in $\mathrm{Wh}(\pi_1 N)$. Then the torsion of $(W\cup_f W',M)$ as measured in $\mathrm{Wh}(\pi_1 N)$ is $\sigma+f_\#\sigma'$. Two special cases of interest are where $W'=W$ (inverted so $W'_0=W_1$, $W'_1=W_0$) and where W' is the inverse of W, i.e. $W\cup W'\simeq M\times I$. In the first case $\sigma'=-\bar{\sigma}$; in the second case $\sigma'=-\sigma$. Thus $(r_M i_N)_\#(\sigma-f_\#\bar{\sigma})$ and $(r_M i_N)_\#(\sigma-f_\#\sigma)$ are in $I(M)$ where $i_N:N\to W$ is the inclusion and $r_M:W\to M$ is a deformation retraction. If $\sigma=\bar{\sigma}$, the two cases coincide. The first case is merely a slight modification of the doubling construction. The second case arises naturally in trying to fiber $N_f=N\times[0,1]/(x,1)\sim(f(x),0)$ over the circle when one splits along M. For this construction to be of any value, there must exist manifolds N and PL homeomorphisms f of N which induce nontrivial maps f_* of $\pi_1 N$ and then $f_\#$ of $\mathrm{Wh}(\pi_1 N)$. Our next proposition concerns f_* when $\pi_1 N=\mathbf{Z}_q$; we only examine $f_\#$ for the special case of $\mathbf{Z}_5$.

PROPOSITION 2. *Let α_r be the automorphism of $\mathbf{Z}_q$ such that $\alpha_r(1)=r$, $(r,q)=1$. Then there is a manifold N (of any given dimension $\geqq 5$) and a* PL *homeomorphism $f:N\circlearrowleft$ with $\pi_1 N=\mathbf{Z}_q$ and $f_*=\alpha_r$.*

PROOF. Let P_q denote the pseudoprojective plane $S^1\cup_q e^2$. Olum [6] proves that there is a simple homotopy equivalence $g:P_q\circlearrowleft$ with $g_*=\alpha_r$ (under the natural isomorphism $\pi_1 P_q\simeq\mathbf{Z}_q$). Embed P_q as a subcomplex of E^{n+1}, $n\geqq 6$ and let N be the boundary of a regular neighborhood R of P_q.

The composition of g with the inclusion of P_q into the interior of R is homotopic to an imbedding h. By [4] and uniqueness of regular neighborhoods, h may be extended to an imbedding k of R into the interior of R. Since g is a simple homotopy equivalence, so is k; excision and the s-cobordism theorem then imply that $R\backslash\operatorname{int} k(R)\simeq N\times[0, 1]$. Using collars, we may modify k to give a PL homeomorphism $l: R\to R$. Letting $f=l|N$, then $f_*=\alpha_r$ (in terms of the isomorphisms $\pi_1 N\simeq\pi_1 R\simeq\pi_1 P_q\simeq \mathbf{Z}_q$).

Thus we may realize any possible automorphism of $\mathbf{Z}_q$. The next question is whether $(\alpha_r)_\#:\mathrm{Wh}(\mathbf{Z}_q)\circlearrowleft$ is ever nontrivial. Since we have a more effective technique for $\mathbf{Z}_q$, we content ourselves with one example: $q=5$. Then $\mathrm{Wh}(\mathbf{Z}_5)\simeq\mathbf{Z}$, generated by the unit $t+t^{-1}-1\in \mathbf{Z}(\mathbf{Z}_5)$, with inverse $t^2+t^{-2}-1$, where t denotes the generator of $\mathbf{Z}_5$ [5]. Thus the automorphism $\alpha_2:\mathbf{Z}_5\circlearrowleft$ has $(\alpha_2)_\#(t+t^{-1}-1)=t^2+t^{-2}-1=(t+t^{-1}-1)^{-1}$. Thus $2\mathrm{Wh}(\mathbf{Z}_5)\subset I(M)$, where $M=W_1(N, [t+t^{-1}-1])$, N as in Proposition 2.

The proof of Proposition 2 suggests another technique which yields our realizability theorem for $\mathbf{Z}_q$ using another result of Olum [6].

PROPOSITION 3. *Let $n\geqq 5$, $X\subset E^{n+1}$ a 2-complex and $g: X\circlearrowleft$ a homotopy equivalence with torsion $\tau\in\mathrm{Wh}(\pi_1 X)$. Then if N is the boundary of a regular neighborhood R of X, we have $F_\#\tau\in I(N)$ where F is the composition of isomorphisms $\pi_1 X\simeq\pi_1 R\simeq\pi_1 N$ induced by inclusions.*

PROOF. Homotope $X\xrightarrow{g}X\subsetneq\operatorname{int} R$ to an imbedding h. h then extends to an imbedding $k: R\to\operatorname{int} R$ by [4]. Then $R\backslash\operatorname{int} k(R)$ is an inertial h-cobordism with boundary $N\cup k(N)$. Using excision, one computes the torsion of $k(N)\subset R\backslash\operatorname{int} k(R)$ as $G_*\tau\in\mathrm{Wh}(\pi_1(k(N)))$ where $G:\pi_1 X\to\pi_1 k(N)$ is the composition $\pi_1 X\xrightarrow{g_*}\pi_1 X\xrightarrow{i_*}\pi_1 k(R)\xrightarrow{j_*^{-1}}\pi_1 k(N)$. Transferring from $k(N)$ to N via $(k|N)^{-1}$ gives $F_\#\tau\in I(N)$ for $F=(k|N)_*^{-1}G$. One then checks F is just the composition of isomorphisms $\pi_1 X\simeq\pi_1 R\simeq\pi_1 N$ induced by inclusions.

COROLLARY 2. *Given $n\geqq 5$, $q\in\mathbf{N}$, there is a manifold N of dimension n with $\pi_1 N\simeq\mathbf{Z}_q$ and* $\mathrm{Wh}(\pi_1 N)=I(N)$.

PROOF. This follows from Proposition 3 with $X=P_q$ using Olum's result [6] that there are self homotopy equivalences of P_q with prescribed torsion in $\mathrm{Wh}(\pi_1 P_q)$.

REMARK. Propositions 2 and 3 make it clear that the realizability problem is really one of CW complexes and (simple) homotopy equivalences.

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