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MAXIMAL SUBLATTICES OF FINITE DISTRIBUTIVE LATTICES. II

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ABSTRACT. Let L be a lattice, $J(L)=\{x \in L|x$ join-irreducible in $L\}$ and $M(L)=\{x \in L|x$ meet-irreducible in $L\}$. As is well known the sets $J(L)$ and $M(L)$ play a central role in the arithmetic of a lattice L of finite length and particularly, in the case that L is distributive. It is shown that the "quotient set" $Q(L)=\{b/a|a \in J(L), b \in M(L), a \leqq b\}$ plays a somewhat analogous role in the study of the sublattices of a lattice L of finite length. If L is a finite distributive lattice, its quotient set $Q(L)$ in a natural way determines the lattice of all sublattices of L. By examining the connection between $J(K)$ and $J(L)$, where K is a maximal proper sublattice of a finite distributive lattice L, the following is proven: every finite distributive lattice of order $n \geqq 3$ which contains a maximal proper sublattice of order m also contains sublattices of orders $n-m$, $2(n-m)$, and $3(n-m)$; and, every finite distributive lattice L contains a maximal proper sublattice K such that either $|K|=|L|-1$ or $|K| \geqq 2l(L)$, where $l(L)$ denotes the length of L.

1. **Introduction.** Let L be a lattice, $J(L)=\{x \in L|x$ join-irreducible in $L\}$ and $M(L)=\{x \in L|x$ meet-irreducible in $L\}$. As is well known the sets $J(L)$ and $M(L)$ play a central role in the arithmetic of a lattice L of finite length and particularly, in the case that L is distributive. We show (Proposition 1) that the "quotient set" $Q(L)=\{b/a|a \in J(L), b \in M(L), a \leqq b\}$ plays a somewhat analogous role in the study of the sublattices of a lattice L of finite length. If L is a finite distributive lattice, its quotient set $Q(L)$ in a natural way determines (Theorem 1) the lattice $\mathrm{Sub}(L)$ of all sublattices of L.

By examining (Theorem 2) the connection between $J(K)$ and $J(L)$, where K is a maximal proper sublattice of a finite distributive lattice L, we can derive some useful information about the orders of sublattices of finite distributive lattices; namely, *every finite distributive lattice of order $n \geqq 3$ which contains a maximal proper sublattice of order m also contains*

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sublattices of orders $n-m$, $2(n-m)$, *and* $3(n-m)$; and, *every finite distributive lattice* L *contains a maximal proper sublattice* K *such that either* $|K|=|L|-1$ *or* $|K|\geqq 2l(L)$, where $l(L)$ denotes the length of L.

The author wishes to thank Barry Wolk for suggesting the proof presented here for Proposition 1. For all terminology not explained here we refer to G. Birkhoff **[1]**.

2. **A connection between** $Q(L)$ **and** Sub(L). Proposition 1 below serves to underline a basic connection between $Q(L)$ and the sublattices of a lattice L of finite length, a connection which, specialized to finite distributive lattices, has been the motivation for the results presented in this paper.

Proposition 1, in fact, is a generalization of Lemma 1 **[2]**. We shall throughout adopt the abbreviation $\bigcup_A [a, b]$ for $\bigcup_{b/a\in A} [a, b]$, where $A\subseteq Q(L)$.

PROPOSITION 1. *If* S *is a sublattice of a lattice* L *of finite length then* $S=L-\bigcup_A [a, b]$, *for some* $A\subseteq Q(L)$.

PROOF. We must show that for every $x\in L-S$ there is some $b/a\in Q(L)$ such that $x\in[a, b]\subseteq L-S$. Let us suppose that this does not hold for some $x\in L-S$. Let $A=\{a\in J(L)\mid a\leqq x\}$ and $B=\{b\in M(L)\mid x\leqq b\}$; clearly, $A\neq\varnothing\neq B$ and $\bigvee A=x=\bigwedge B$. But then by our assumption, for every $a\in A$ and for every $b\in B$ there exists $y_a^b\in S\cap[a, b]$. Since L is of finite length it is complete; therefore, $\bigvee_{a\in A}\bigwedge_{b\in B} y_a^b\in S$. On the other hand,

$$x=\bigvee A=\bigvee_{a\in A}\bigwedge_{b\in B} a\leqq\bigvee_{a\in A}\bigwedge_{b\in B} y_a^b\leqq\bigvee_{a\in A}\bigwedge_{b\in B} b=\bigwedge B=x,$$

that is, $\bigvee_{a\in A}\bigwedge_{b\in B} y_a^b=x\in L-S$, which is a contradiction.

In view of Proposition 1 it is natural to classify sublattices of a lattice L of finite length in terms of subsets of $Q(L)$. Indeed, for $A\subseteq Q(L)$ we define $\mathrm{Cl}(A)=\{y/x\in Q(L)\mid [x, y]\subseteq\bigcup_A [a, b]\}$ and $\mathrm{Cl}(Q(L))=\{\mathrm{Cl}(A)\mid A\subseteq Q(L)\}$.

The following lemma is straightforward.

LEMMA 1. *Let* L *be a lattice of finite length and* $A, B\subseteq Q(L)$. *Then*
(i) $\bigcup_A [a, b]=\bigcup_{\mathrm{Cl}(A)} [x, y]$ *and*,
(ii) $\bigcup_{\mathrm{Cl}(A)} [x, y]\subseteq\bigcup_{\mathrm{Cl}(B)} [u, v]$ *if and only if* $\mathrm{Cl}(A)\subseteq\mathrm{Cl}(B)$.

The next lemma is an easy consequence of Lemma 1.

LEMMA 2. *Let* L *be a lattice of finite length. Then*
(i) Cl *is a closure operator on* $Q(L)$ *and*,
(ii) $\mathrm{Cl}(Q(L))$ *is a lattice with respect to set inclusion.*

THEOREM 1. *For a lattice L of finite length the following conditions are equivalent*:

(i) *L is distributive*;

(ii) *$L-\bigcup_A [a, b]$ is a sublattice of L for every $A\subseteq Q(L)$*;

(iii) *for every $S\subseteq L$, S is a sublattice of L if and only if $S=L-\bigcup_A [a, b]$ for some $A\subseteq Q(L)$*;

(iv) *the mapping $\varphi(S)=\mathrm{Cl}(A)$, where $S=L-\bigcup_A [a, b]$, $A\subseteq Q(L)$, is an isomorphism between* $\mathrm{Sub}(L)$ *and the dual of* $\mathrm{Cl}(Q(L))$.

PROOF. That (i) implies (ii) follows from the fact that join-irreducible elements in a distributive lattice are *join-prime*, that is, if $a\in J(L)$ and $a\leqq b\vee c$ then $a\leqq b$ or $a\leqq c$. Applying Proposition 1 we get that (ii) implies (iii). On the other hand, Proposition 1 together with Lemma 1(ii) shows that φ is well-defined, one-one, isotone, and that, in fact, φ^{-1} is isotone. From (iii) we have that φ is onto, so that φ is, indeed, an isomorphism; thus, (iii) implies (iv). It remains only to show that (iv) implies (i).

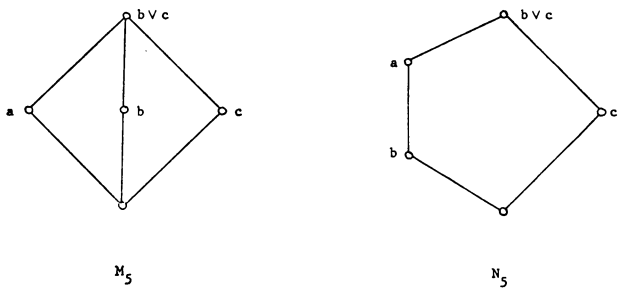

FIGURE 1

Let M_5 and N_5 be the two five-element nondistributive lattices labelled as in Figure 1. Suppose that L satisfies (iv) but L is nondistributive. Then L contains as a sublattice a copy of M_5 or N_5. Let d be a join-irreducible in L such that $d\leqq a$ but $d\nleqq b\wedge c$, and e a meet-irreducible in L such that $e\geqq b\vee c$. By the surjectivity of φ^{-1}, $L-\bigcup_{\mathrm{Cl}(\{e/d\})} [x, y]$ is a sublattice of L. In view of Lemma 1(i), $L-\bigcup_{\mathrm{Cl}(\{e/d\})} [x, y]=L-[d, e]$. But $b\vee c\in [d, e]$ although $b, c\in L-[d, e]$, which is a contradiction. Thus, (iv) implies (i), completing the proof.

3. **Maximal proper sublattices of finite distributive lattices.** We define a partial ordering on $Q(L)$ as follows: $b/a\leqq d/c$ *if and only if* $[a, b]\subseteq [c, d]$. If b/a is minimal with respect to this ordering then $\mathrm{Cl}(\{b/a\})=\{b/a\}$ so that by Theorem 1, $L-[a, b]$ is a maximal proper sublattice of L in the case that L is finite distributive. Note that if $b/a\in Q(L)$ then b/a is minimal if and only if $[a, b)\subseteq L-M(L)$ and $(a, b]\subseteq L-J(L)$ (cf. [**2**, Theorem 3]).

For $x, y \in L$, *x covers y* ($x \succ y$ or $y \prec x$) *in L* if $x > y$ and $x \geqq z > y$ implies $x = z$, for every $z \in L$. For $A \subseteq L$ we define $\mathrm{cov}(A) = \{x \in L \mid x \succ a$ or $x \prec a$ or $x = a$, for some $a \in A\}$. Observe that $a \in L - J(L)$ ($a \in L - M(L)$) if and only if there exist $b, c \in \mathrm{cov}(\{a\})$ such that $a = b \vee c$ ($a = b \wedge c$).

THEOREM 2. *Let L be a finite distributive lattice and $K = L - [a, b]$ ($b/a \in Q(L)$, $a \neq b$) be a maximal proper sublattice of L. Then* (i) $\mathrm{cov}([a, b])$ *is a sublattice of L isomorphic to the direct product of* $[a, b]$ *with a three-element chain, and* (ii) $J(K) = (J(L) - \{a\}) \cup \{c\}$, *where* $a \prec c \in K$.

PROOF. Set

$$\begin{aligned} A &= \{y \in K \mid y \prec x \text{ for some } x \in [a, b]\}, \\ B &= \{y \in K \mid y \succ x \text{ for some } x \in [a, b]\}, \\ A' &= \{x \in [a, b] \mid x \succ y \text{ for some } y \in A\}, \\ B' &= \{x \in [a, b] \mid x \prec y \text{ for some } y \in B\}. \end{aligned}$$

To establish (i) it suffices to show that $A \simeq [a, b] \simeq B$. Since b/a is minimal in $Q(L)$ and $a \neq b$, it follows that $a \neq 0$ and $b \neq 1$; thus, $a \in A'$ and $b \in B'$. Furthermore, since $L - [a, b]$ is a sublattice of L, every element in A' covers precisely one element in A and every element in B' is covered by precisely one element in B.

Suppose now that c_1', c_2' are distinct minimal elements in B' with covers $c_1, c_2 \in B$. Since $[a, b]$ is a sublattice of L, $c_1 \neq c_2$; since c_1' is incomparable with c_2', c_1 is incomparable with c_2; and since $L - [a, b]$ is a sublattice of L, $c_1, c_2 \succ c_1 \wedge c_2 \in L - [a, b]$. Now, if $c_2' = c_2' \vee (c_1 \wedge c_2)$ then $c_1' \wedge c_2' < c_1 \wedge c_2 < c_2'$, so that $c_1 \wedge c_2 \in [a, b]$. Therefore, $c_2' < c_2' \vee (c_1 \wedge c_2) \leqq c_2$ and, since $c_2' \prec c_2$, we have that $c_2' \prec c_2' \vee (c_1 \wedge c_2) = c_2$ which, by transposition implies that $c_2' \wedge (c_1 \wedge c_2) \prec c_1 \wedge c_2$. But $c_1' \wedge c_2' \leqq c_2' \wedge (c_1 \wedge c_2) < c_2'$ so that $c_2' \wedge (c_1 \wedge c_2) \in B'$, contradicting the minimality of c_2'. Thus, B' has a unique minimal element c' with precisely one cover c in B; dually, A' has a unique maximal element d' covering precisely one element d in A. Now, if f is the unique cover of b and e the unique element covered by a then by transposition we have that $A = [e, d] \simeq [a, d'] = A'$ and $B = [c, f] \simeq [c', b] = B'$. From this it follows that $\mathrm{cov}([a, b]) = A \cup [a, b] \cup B$ is a sublattice of L and that in fact, b/a is minimal in $Q(\mathrm{cov}([a, b]))$. In this case $A \cup B$ is a maximal proper sublattice of $\mathrm{cov}([a, b])$ so that by [**2**, Theorem 2], $|A \cup B| \geqq \frac{2}{3}|\mathrm{cov}([a, b])|$. Now, if $d < b$ or $a < c$ then $|\mathrm{cov}([a, b])| = |A| + |[a, b]| + |B| < 3|[a, b]|$. But $[a, b] = \mathrm{cov}([a, b]) - (A \cup B)$ so that $|A \cup B| < \frac{2}{3}|\mathrm{cov}([a, b])|$, which is a contradiction. Thus, $a = c'$ and $b = d'$ so that $A \simeq [a, b] \simeq B$, from which (i) follows.

To show (ii) observe first that $J(L) - \{a\} \subseteq J(K)$ and $J(K) \cap A \subseteq J(L) - \{a\}$. It suffices then to show that $J(K) \cap B = \{c\}$.

Let $x \in B-\{c\}$. Choose some $y \in B$ such that $x \succ y$. Then there exist $x_1, y_1 \in [a, b]$ and $x_2, y_2 \in A$ such that $x \succ x_1 \succ x_2$ and $y \succ y_1 \succ y_2$. By transposition $x_1 \succ y_1$, $x_2 \succ y_2$, and $x_1 \wedge y = y_1$. If $x_2 < y$ then $y_1 = x_1 \wedge y \geqq x_2 \succ y_2$, and since $y_1 \succ y_2$ we have that $y_1 = x_2$, which is impossible. Thus, x_2 is incomparable with y and, in fact, x covers x_2 in K, and since x also covers y in K, we get that x is join-reducible in K.

It remains only to show that $c \in J(K)$. We may without loss of generality assume that c covers two distinct incomparable elements $c_1, c_2 \in L$, both incomparable with a. But a is join-irreducible in L, that is, it covers only e. By transposition we get that $\{a, c_1, c_2, e, c\}$ is a sublattice of L isomorphic to the five-element modular, nondistributive lattice M_5 which, of course, is a contradiction. The proof of the theorem is now complete.

The following corollary is an immediate consequence of Theorem 2(i).

COROLLARY 1. *Every distributive lattice of order $n \geqq 3$ which contains a maximal proper sublattice of order m also contains sublattices of orders $n-m$, $2(n-m)$, and $3(n-m)$.*

COROLLARY 2. *Every finite distributive lattice L contains a maximal proper sublattice K such that either $|K|=|L|-1$ or $|K| \geqq 2l(L)$.*

PROOF. We may without loss of generality assume that $\mathrm{Irr}(L)=\varnothing$. Recall that for finite distributive lattices $|J(L)|=l(L)+1=|M(L)|$. Furthermore, the inequality $|L| \geqq |J(L)|+|M(L)|-|\mathrm{Irr}(L)|$ holds in every lattice L of finite length, so that if L is distributive we have that $|L| \geqq 2(l(L)+1)-|\mathrm{Irr}(L)|$. (This latter inequality, incidentally, holds in every lattice of finite length, cf. [**3**, Theorem 1].)

If $J(K)=J(L)-\{a\}$ then $M(K)=M(L)-\{b\}$, and since $J(L) \cap M(L) = \mathrm{Irr}(L)=\varnothing$ we also have that $\mathrm{Irr}(K)=\varnothing$. In this case $|K| \geqq 2(l(K)+1)-|\mathrm{Irr}(K)|=2|J(L)-\{a\}|=2l(L)$.

Otherwise, $J(K) \neq J(L)-\{a\}$. By Theorem 2(ii) and its dual there exist $c, d \in L$ such that $J(K)=(J(L)-\{a\}) \cup \{c\}$, $c \notin J(L)$, and $M(K(=(M(L)-\{b\}) \cup \{d\}$, $d \notin M(L)$. Observe that $(J(L)-\{a\}) \cap (M(L)-\{b\}) \subseteq J(L) \cap M(L)=\mathrm{Irr}(L)=\varnothing$. Therefore, $\mathrm{Irr}(K) \subseteq \{c, d\}$, so that in this case

$$\begin{aligned} |K| &\geqq 2(l(K)+1)-|\mathrm{Irr}(K)| \\ &\geqq 2(|(J(L)-\{a\}) \cup \{c\}|+1)-2 = 2l(L). \end{aligned}$$

The estimate on the order of maximal proper sublattices of finite distributive lattices prescribed in Corollary 2 is best possible in the sense that, if for every positive integer n, L_n is the ordinal sum of n copies of the Boolean lattice $\mathbf{2}^3$, then the *maximum* order of a maximal proper sublattice of L_n is $2l(L_n)$.

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