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ON L_p-SPECTRA OF THE LAPLACIAN ON A LIE GROUP WITH POLYNOMIAL GROWTH

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ABSTRACT. The following theorem is proved: *If G is a Lie group with polynomial growth (a compact extension of a nilpotent group, e.g.) and $\Delta=X_1^2+\cdots+X_n^2$, where $X_1, \cdots, X_n$ is a basis of the Lie algebra of G, then for all p, $1\leqq p<\infty$, the operator Δ has the same spectrum on all $L_p(G)$.*

Let G be a locally compact group. For a Borel subset M of G, the left invariant Haar measure of M is denoted by $|M|$. We write M^n for $\{g_1\cdots g_n: g_j\in M\}$.

We say that G has polynomial growth if there is a number r such that for every compact subset M of G we have $|M^n|=O(n^r)$ as $n\to\infty$.

Among the connected Lie groups the nilpotent groups and their compact extensions are of polynomial growth while most of the solvable groups are not (cf. [**1**], [**6**]). Groups of polynomial growth are unimodular.

The aim of this note is to use the results of [**5**] to compare the spectra of the laplacian considered on various $L_p(G)$, $1\leqq p<\infty$, for a Lie group G. The result is that all the spectra are equal provided G has polynomial growth.

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Let G be a Lie group, LG the Lie algebra of G whose elements are viewed as differential operators of the first order on $C_0^\infty(G)$ which commute with the right translations. Let $X_1, \cdots, X_n$ be a basis of LG and let $\Delta=X_1^2+\cdots+X_n^2$. For every p, $1\leqq p<\infty$, the operator Δ is densely defined on $L_p(G)$ and admits the closure; moreover, it is essentially selfadjoint on $L_2(G)$ (cf. [**2**], [**7**], [**8**]).

For an operator A whose domain is dense in all $L_p(G)$, $1\leqq p<\infty$, let

$$\sigma_p(A)=\{\lambda\in \mathbf{C}:(\lambda-A)^{-1} \text{ is bounded in } L_p(G)\}^c.$$

We prove the following:

THEOREM. *If G is a Lie group of polynomial growth, then $\sigma_p(\Delta)=\sigma_2(\Delta)$ for all $1\leqq p<\infty$.*

We start with a lemma.

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LEMMA. *Let M be a measure space and let X be a dense subspace of all $L_p(M)$, $1 \leqq p < \infty$. Suppose A is an operator defined on X such that $AX \subset X$ and A is essentially selfadjoint on $L_2(M)$. If $\sigma_1(A) = \sigma_2(A)$, then $\sigma_p(A) = \sigma_2(A)$ for all $1 \leqq p < \infty$.*

PROOF. Suppose $\lambda \notin \sigma_1(A) = \sigma_2(A)$. Then $R_\lambda = (\lambda - A)^{-1}$ is bounded on both $L_1(M)$ and $L_2(M)$, and so by the Riesz-Thorin interpolation theorem, R_λ is bounded on $L_p(M)$, $1 < p \leqq 2$. Hence $\sigma_p(A) \subset \sigma_2(A)$ for $1 < p \leqq 2$. This shows that $\sigma_p(A)$ is real. Since A is symmetric on $L_2(M)$,

$$Af = A'f \quad \text{for } f \in X, \tag{1}$$

and so A admits a closure on all $L_p(M)$, $1 < p < \infty$. Consequently, (cf., e.g., [10, VIII, 6]), $\sigma_p(A) = \sigma_q(A')$, where $q = p(p-1)^{-1}$. Hence, by (1), we have

$$\sigma_p(A) = \sigma_q(A), \qquad 1 < p \leqq 2. \tag{2}$$

Since, if $1 < p \leqq 2$, then $2 \leqq q < \infty$, (2) together with the Riesz-Thorin theorem gives

$$\sigma_2(A) \subset \sigma_p(A) = \sigma_q(A),$$

which completes the proof of the lemma.

PROOF OF THEOREM. Let $\{T^t\}_{t \in R^+}$ denote the diffusion semigroup (cf. [9]) defined on all $L_p(G)$, $1 \leqq p < \infty$. On $L_2(G)$, T^t can be written in the form $T^t = e^{t\Delta}$. We have

$$T^t f = p_t^* f, \qquad f \in L_p(G),$$

where $p_t \in L_1(G)$, p_t is real, nonnegative, $p_t = p_t^*$, where * denotes the involution in $L_1(G)$ (cf. [2], [8]). Moreover, p_t is a rapidly decreasing function in the sense of [5], i.e.

$$\int_{G \setminus M^n} p_t(g)\, dg = o(n^{-k}) \quad \text{as } n \to \infty,$$

for all compact neighborhoods of the identity M and all natural numbers k (cf. [2] and [7]).

Consequently, by virtue of [5], the spectrum of p_t in the Banach*-algebra $L_1(G)$ is equal to $\sigma_2(T^t)$, which is clearly equivalent to the equality $\sigma_1(T^t) = \sigma_2(T^t)$ for all $t \in R^+$. Hence, by the Lemma, for all $t \in R^+$ and $1 \leqq p < \infty$,

$$\sigma_p(T^t) = \sigma_2(T^t). \tag{3}$$

It follows from [4, Corollary 2, p. 457] that $\exp\{t\sigma_p(\Delta)\} \subset \sigma_p(T^t) = \sigma_2(e^{t\Delta})$ for all $t \in R^+$. But, by the spectral theorem, since Δ is nonpositive,

$$\sigma_2(e^{t\Delta}) = \exp\{t\sigma_2(\Delta)\}^- = \exp\{t\sigma_2(\Delta)\} \cup \{0\}.$$

Thus

$$\exp\{t\sigma_p(\Delta)\} \subset \exp\{t\sigma_2(\Delta)\} \quad \text{for all } t > 0,$$

whence, since $t\sigma_2(\Delta)$ is real for all $t>0$,

$$\sigma_p(\Delta) \subset \sigma_2(\Delta), \qquad 1 \leqq p < \infty. \tag{4}$$

Suppose now that $\lambda \notin \sigma_1(\Delta)$. Then $R_\lambda = (\lambda - \Delta)^{-1}$ is a bounded operator on $L_1(G)$ which commutes with the right translations, since Δ does. Therefore R_λ is a multiplier on $L_1(G)$ and so $R * f = \mu * f$ for f in $L_1(G)$, where μ is a bounded measure (cf. [3, Theorem 35.5, p. 376]). Therefore R_λ is bounded on all $L_p(G)$, $1 \leqq p < \infty$, and consequently, $\lambda \notin \sigma_p(\Delta)$. This, together with (4), implies

$$\sigma_1(\Delta) = \sigma_2(\Delta)$$

and, by the lemma, completes the proof of the Theorem.

References

1. J. Dixmier, *Opérateurs de rang fini dans les représentations unitaires*, Inst. Hautes Études Sci. Publ. Math. No. 6 (1960), 13–25. MR **25** #149.

2. L. Gårding, *Vecteurs analytiques dans les représentations des groupes de Lie*, Bull. Soc. Math. France **88** (1960), 73–93. MR **22** #9870.

3. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. Vol. II. *Structure and analysis for compact groups analysis on locally compact Abelian groups*, Die Grundlehren der math. Wissenschaften, Band 152; Springer-Verlag, New York and Berlin, 1970. MR **41** #7378.

4. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, R.I., 1957. MR **19**, 664.

5. A. Hulanicki, *On the spectrum of convolution operators on groups with polynomial growth*, Invent. Math. **17** (1972), 135–142.

6. J. W. Jenkins, *A characterization of growth in locally compact groups*, J. Functional Analysis **12** (1973), 113–127.

7. E. Nelson, *Analytic vectors*, Ann. of Math. (2) **70** (1959), 572–615. MR **21** #5901.

8. E. Nelson and W. F. Stinspring, *Representation of elliptic operators in an enveloping algebra*, Amer. J. Math. **81** (1959), 547–560. MR **22** #907.

9. E. M. Stein, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. of Math. Studies, no. 63, Princeton Univ. Press, Princeton, N.J.; Univ. of Tokyo Press, Tokyo, 1970. MR **40** #6176.

10. K. Yosida, *Functional analysis*, Die Grundlehren der math. Wissenschaften, Band 123, Academic Press, New York; Springer-Verlag, Berlin, 1965. MR **31** #5054.

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