

## ON $L_p$ -SPECTRA OF THE LAPLACIAN ON A LIE GROUP WITH POLYNOMIAL GROWTH

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**ABSTRACT.** The following theorem is proved: *If  $G$  is a Lie group with polynomial growth (a compact extension of a nilpotent group, e.g.) and  $\Delta = X_1^2 + \cdots + X_n^2$ , where  $X_1, \dots, X_n$  is a basis of the Lie algebra of  $G$ , then for all  $p$ ,  $1 \leq p < \infty$ , the operator  $\Delta$  has the same spectrum on all  $L_p(G)$ .*

Let  $G$  be a locally compact group. For a Borel subset  $M$  of  $G$ , the left invariant Haar measure of  $M$  is denoted by  $|M|$ . We write  $M^n$  for  $\{g_1 \cdots g_n : g_j \in M\}$ .

We say that  $G$  has polynomial growth if there is a number  $r$  such that for every compact subset  $M$  of  $G$  we have  $|M^n| = O(n^r)$  as  $n \rightarrow \infty$ .

Among the connected Lie groups the nilpotent groups and their compact extensions are of polynomial growth while most of the solvable groups are not (cf. [1], [6]). Groups of polynomial growth are unimodular.

The aim of this note is to use the results of [5] to compare the spectra of the laplacian considered on various  $L_p(G)$ ,  $1 \leq p < \infty$ , for a Lie group  $G$ . The result is that all the spectra are equal provided  $G$  has polynomial growth.

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Let  $G$  be a Lie group,  $LG$  the Lie algebra of  $G$  whose elements are viewed as differential operators of the first order on  $C_0^\infty(G)$  which commute with the right translations. Let  $X_1, \dots, X_n$  be a basis of  $LG$  and let  $\Delta = X_1^2 + \cdots + X_n^2$ . For every  $p$ ,  $1 \leq p < \infty$ , the operator  $\Delta$  is densely defined on  $L_p(G)$  and admits the closure; moreover, it is essentially selfadjoint on  $L_2(G)$  (cf. [2], [7], [8]).

For an operator  $A$  whose domain is dense in all  $L_p(G)$ ,  $1 \leq p < \infty$ , let

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : (\lambda - A)^{-1} \text{ is bounded in } L_p(G)\}^c.$$

We prove the following:

**THEOREM.** *If  $G$  is a Lie group of polynomial growth, then  $\sigma_p(\Delta) = \sigma_2(\Delta)$  for all  $1 \leq p < \infty$ .*

We start with a lemma.

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LEMMA. Let  $M$  be a measure space and let  $X$  be a dense subspace of all  $L_p(M)$ ,  $1 \leq p < \infty$ . Suppose  $A$  is an operator defined on  $X$  such that  $A^2X \subset X$  and  $A$  is essentially selfadjoint on  $L_2(M)$ . If  $\sigma_1(A) = \sigma_2(A)$ , then  $\sigma_p(A) = \sigma_q(A)$  for all  $1 \leq p < \infty$ .

PROOF. Suppose  $\lambda \notin \sigma_1(A) = \sigma_2(A)$ . Then  $R_\lambda = (\lambda - A)^{-1}$  is bounded on both  $L_1(M)$  and  $L_2(M)$ , and so by the Riesz-Thorin interpolation theorem,  $R_\lambda$  is bounded on  $L_p(M)$ ,  $1 < p \leq 2$ . Hence  $\sigma_p(A) \subset \sigma_2(A)$  for  $1 < p \leq 2$ . This shows that  $\sigma_p(A)$  is real. Since  $A$  is symmetric on  $L_2(M)$ ,

$$(1) \quad Af = A'f \quad \text{for } f \in X,$$

and so  $A$  admits a closure on all  $L_p(M)$ ,  $1 < p < \infty$ . Consequently, (cf., e.g., [10, VIII, 6]),  $\sigma_p(A) = \sigma_q(A')$ , where  $q = p(p-1)^{-1}$ . Hence, by (1), we have

$$(2) \quad \sigma_p(A) = \sigma_q(A), \quad 1 < p \leq 2.$$

Since, if  $1 < p \leq 2$ , then  $2 \leq q < \infty$ , (2) together with the Riesz-Thorin theorem gives

$$\sigma_2(A) \subset \sigma_p(A) = \sigma_q(A),$$

which completes the proof of the lemma.

PROOF OF THEOREM. Let  $\{T^t\}_{t \in R^+}$  denote the diffusion semigroup (cf. [9]) defined on all  $L_p(G)$ ,  $1 \leq p < \infty$ . On  $L_2(G)$ ,  $T^t$  can be written in the form  $T^t = e^{t\Delta}$ . We have

$$T^t f = p_t^* f, \quad f \in L_p(G),$$

where  $p_t \in L_1(G)$ ,  $p_t$  is real, nonnegative,  $p_t = p_t^*$ , where  $*$  denotes the involution in  $L_1(G)$  (cf. [2], [8]). Moreover,  $p_t$  is a rapidly decreasing function in the sense of [5], i.e.

$$\int_{G \setminus M^n} p_t(g) dg = o(n^{-k}) \quad \text{as } n \rightarrow \infty,$$

for all compact neighborhoods of the identity  $M$  and all natural numbers  $k$  (cf. [2] and [7]).

Consequently, by virtue of [5], the spectrum of  $p_t$  in the Banach\*-algebra  $L_1(G)$  is equal to  $\sigma_2(T^t)$ , which is clearly equivalent to the equality  $\sigma_1(T^t) = \sigma_2(T^t)$  for all  $t \in R^+$ . Hence, by the Lemma, for all  $t \in R^+$  and  $1 \leq p < \infty$ ,

$$(3) \quad \sigma_p(T^t) = \sigma_2(T^t).$$

It follows from [4, Corollary 2, p. 457] that  $\exp\{t\sigma_p(\Delta)\} \subset \sigma_p(T^t) = \sigma_2(e^{t\Delta})$  for all  $t \in R^+$ . But, by the spectral theorem, since  $\Delta$  is nonpositive,

$$\sigma_2(e^{t\Delta}) = \exp\{t\sigma_2(\Delta)\}^- = \exp\{t\sigma_2(\Delta)\} \cup \{0\}.$$

Thus

$$\exp\{t\sigma_p(\Delta)\} \subset \exp\{t\sigma_2(\Delta)\} \quad \text{for all } t > 0,$$

whence, since  $t\sigma_2(\Delta)$  is real for all  $t > 0$ ,

$$(4) \quad \sigma_p(\Delta) \subset \sigma_2(\Delta), \quad 1 \leq p < \infty.$$

Suppose now that  $\lambda \notin \sigma_1(\Delta)$ . Then  $R_\lambda = (\lambda - \Delta)^{-1}$  is a bounded operator on  $L_1(G)$  which commutes with the right translations, since  $\Delta$  does. Therefore  $R_\lambda$  is a multiplier on  $L_1(G)$  and so  $R * f = \mu * f$  for  $f$  in  $L_1(G)$ , where  $\mu$  is a bounded measure (cf. [3, Theorem 35.5, p. 376]). Therefore  $R_\lambda$  is bounded on all  $L_p(G)$ ,  $1 \leq p < \infty$ , and consequently,  $\lambda \notin \sigma_p(\Delta)$ . This, together with (4), implies

$$\sigma_1(\Delta) = \sigma_2(\Delta)$$

and, by the lemma, completes the proof of the Theorem.

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