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CHARACTERIZATION OF MERGELYAN SETS

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ABSTRACT. Necessary and sufficient conditions on a relatively closed subset F of $D=\{z:|z|<1\}$ are given such that each analytic function in D which is uniformly continuous on F can be uniformly approximated by polynomials on $K\cup F$ for each compact subset K of D.

Let $D=\{z:|z|<1\}$ and let F be a relatively closed subset. $U(F)$ denotes all analytic functions in D whose restriction to F is uniformly continuous. F is called a Mergelyan set if each $f\in U(F)$ can be uniformly approximated by polynomials on $K\cup F$ for each compact subset K of D.

The aim of this paper is to characterize Mergelyan sets. The author wishes to thank Professor L. A. Rubel for suggesting this problem and for some very helpful comments. Some results about Mergelyan sets can be found in [**4**].

Before stating our main result we need some notation. Let $H(F)$ denote all analytic functions in D which are bounded on F. We say that the polynomials are dense in $H(F)$ if each $f\subseteq H(F)$ is a uniform limit on compact subsets of D of a sequence of polynomials $\{p_n\}$, $\{p_n\}$ being uniformly bounded on F. If $\{p_n\}$ can be chosen such that $\sup\{|p_n(z)|, z\in F\}\leqq \sup\{|f(z)|, z\in F\}+1/n$ the polynomials are said to be strongly dense in $H(F)$.

Whenever B is a nonempty subset of the complex plane C and f is a (complex valued) function on B we put $\|f\|_B=\sup\{|f(z)|, z\in B\}$. If $B\subset S\subset C$ and A is a family of functions on S we define $\hat{B}^A=\bigcap_{f\in A}\{z\in S: |f(z)|\leqq \|f\|_B\}$. If $S=C$ and A is the set of all polynomials we write $\hat{B}^A=\hat{B}$. As usual $\bar{B}$, B° and ∂B denote the closure, interior and the boundary of B, respectively. By a "measure" we mean a regular Borel measure with compact support in C. The total variation of μ is denoted by $|\mu|$, and if $B\subset C$ is a Borel set, μ_B denotes the restriction of μ to B, i.e. $\mu_B(B')=\mu(B\cap B')$ for each Borel set B'. Finally $C(B)$ denotes all continuous functions on B.

With the notation as above we characterize Mergelyan sets in the following way.

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THEOREM 1. *Let $F\subset D$ be relatively closed. The following statements are equivalent*:

(i) *F is a Mergelyan set.*

(ii) *$D\cap(K\cup F)^{\wedge}=(K\cup F)^{\wedge U(F)}$ whenever $K\subset D$ is compact.*

(iii) *The polynomials are strongly dense in $H(F)$.*

(iv) *The polynomials are dense in $H(F)$.*

(v) *$D\cap(K\cup F)^{\wedge}=(K\cup F)^{\wedge H(F)}$ whenever $K\subset D$ is compact.*

That (ii)$\Leftrightarrow$(v) follows from a geometrical characterization of $(K\cup F)^{\wedge U(F)}$ which we give in Theorem 2. To state Theorem 2 we need a definition.

DEFINITION. Let $U\subset C$ be open and assume $E\subset\partial U$ is nonempty. E is said to be accessible from U if there exists $\{z_n\}_{n=1}^{\infty}\subset U$ and polygonal arcs $\gamma_n\subset U$ connecting z_n to z_{n+1} such that if $K\subset\bar{U}\backslash E$ is compact we have $\gamma_n\cap K=\varnothing$ for n sufficiently large.

THEOREM 2. *Let $S\subset D$ be relatively closed. Then $\hat{S}^{U(S)}=\hat{S}^{H(S)}$. Let V be a bounded component of $C\backslash\bar{S}$. If $V\not\subset\hat{S}^{U(S)}$ we have $V\cap\hat{S}^{U(S)}=\varnothing$ and this happens if and only if $(\partial V)\cap\partial D$ is accessible from V.*

We first prove Theorem 1 assuming Theorem 2.

That (i)$\Rightarrow$(ii) is easy. Let $K\subset D$ be compact and choose $f\in U(F)$ and $z\in D\cap(F\cup K)^{\wedge}$. By hypothesis there exist polynomials p_n converging uniformly to f on $\{z\}\cup F\cap K$. Since $|p_n(z)|<\|p_n\|_{F\cup K}$ for all n we obtain $|f(z)|\leqq\|f\|_{F\cup K}$ by taking limits. Hence $D\cap(F\cup K)^{\wedge}\subset(F\cup K)^{\wedge U(F)}$ and since the opposite inclusion is trivial (ii) follows.

To prove that (ii)$\Rightarrow$(i) we apply some basic results from the theory of uniform algebras. A convenient reference is [**2**]. Let $K\subset D$ be compact and put $X=K\cup\bar{F}$. Denote by $P(X)$ all functions on X which can be uniformly approximated there by polynomials. Denote by A_1 all continuous functions on X which can be uniformly approximated on $K\cup F$ by functions in $U(F)$. If $f\in A_1$ and $\{f_n\}\subset U(F)$ converges uniformly to f on $K\cup F$ it follows by (ii) that $\{f_n\}$ is a Cauchy sequence on $D\cap\hat{X}$ so the functions in A_1 extend in a natural way to functions on $\hat{X}$ which are analytic in $(\hat{X})^{\circ}$. By Mergelyan's theorem [**2**, p. 48] it is sufficient to show that the functions in A_1 are continuous on $\hat{X}$. It is an immediate consequence of the maximum principle for analytic functions that each $f\in A_1$ attains its maximum on $Y=\partial\hat{X}$. If m is a multiplicative functional on A_1 there exists, therefore, a measure μ on Y such that $m(f)=\int f\,d\mu$ for all $f\in A_1$. Let $\pi(m)$ denote the restriction of m to $P(X)$. Since $P(X)$ is a Dirichlet algebra on Y, μ is the unique representing measure for $\pi(m)$ w.r.t. $P(X)$ so the map $m\to\pi(m)$ is 1-1 and by (ii) the map is onto $\hat{X}$. But then the maximal ideal space of A is homeomorphic to $\hat{X}$ so each $f \in A$ is continuous on $\hat{X}$. (The relevant results about Dirichlet algebras and representing measures are in [**2**, pp. 31–37]. The topology of the maximal ideal space is described on pp. 3–4.)

That (iii)$\Rightarrow$(iv) is trivial and we now prove (iv)$\Rightarrow$(v). Assume (iv) and that (v) fails for some compact $K \subset D$. By (iv) it follows that each $f \in H(F)$ must be bounded on $D \cap (K \cup F)^{\wedge}$. This is seen in the same way as we proved (i)$\Rightarrow$(ii). By the maximum principle for analytic functions we can find $f \in H(F)$ and $\{z_n\} \subset D \cap (F \cup K)^{\wedge}$ such that $\|f\|_{F \cup K} < 1$, $2 \leqq |f(z_n)| \leqq 3$ for all n and we can also assume $z_n \to y \in \partial D$ as $n \to \infty$. Let r_n be integers s.t. $r_n \geqq n$ and $(2^{r_n - 1} - \sum_{k=1}^{n-1} 3^{r_k}) \to \infty$ as $n \to \infty$. By an easy induction argument we can construct a subsequence $\{w_n\}_{n=1}^{\infty}$ of $\{z_n\}$ and polynomials $\{p_n\}_{n=1}^{\infty}$ with the following properties for $n = 1, 2, \cdots$:

(1) $|w_{n+1}| > 2^{-1}(1 + |w_n|)$ and $|p_n(w_{n+1})| \geqq 2^{-1}$;

(2) $\|p_n\|_D = 1 = p_n(y)$;

(3) $|p_n(z)(f(z))^{r_n}| < 2^{-n}$ if $|z| < 2^{-1}(1 + |w_n|)$.

We define $h = \sum_1^{\infty} p_n f^{r_n}$. Since $\|f\|_F < 1$ we get from (2) and (3) that $h \in H(F)$. But from the way $\{r_n\}$ was chosen it follows that $|h(w_n)| \to \infty$ as $n \to \infty$ and this contradicts (iv). Hence (v) follows from (iv).

Modulo Theorem 2 the proof of Theorem 1 will be complete if we can show that (v)$\Rightarrow$(iii). Let now $K = \{z : |z| \leqq r < 1\}$ for some fixed r. Define $X = (F \cup K)^{\wedge}$. It is sufficient to show that if $f \in H(F)$, $K_1 \subset \{z : |z| < r\}$ is compact and $\varepsilon > 0$, we can find $f_1 \in P(X)$ s.t.

(4) $\|f_1\|_F \leqq \|f\|_F + \varepsilon$;

(5) $\|f_1 - f\|_{K_1} < \varepsilon$.

As soon as this is done we approximate f_1 uniformly on X by polynomials and (iii) will follow.

We first treat the special case where $\|f\|_F = 0$. Unless $f \equiv 0$, F must be countable and, by (v), $\bar{F} \setminus F$ must be a proper subset of ∂D. In this case $X = K \cup \bar{F}$ and we define first $f_1 = f$ on K and then extend it to be continuous on X and satisfying (4).

We can now assume $\|f\|_F = \eta > 0$ and $\|f\|_{X \cap D} = 1$. Let p_n, $n = 1, 2, \cdots$, be polynomials converging uniformly to f on compact subsets of X° and satisfying $\|p_n\|_X \leqq 1$ for all n. (See [**3**] or [**2**, Theorem 11.1, p. 226].) For each component W_j of X° let λ_j denote harmonic measure on ∂W_1 representing some fixed point in W_j and define $\mu = \sum_j 2^{-j} \lambda_j$ (see [**0**]). Let f^* denote a w^* cluster point of $\{p_n\}$ in $L^{\infty}(\mu)$. Then the harmonic extension of f^* to X° equals f and if $g \in L^{\infty}(\mu)$ has the same property it is well known that $g = f^*$ a.e. $d\mu$ [**2**, Theorem 11.1, p. 226]. We claim that $|f^*| \leqq \eta$ a.e. on $\bar{F} \setminus X^{\circ}$ w.r.t. μ. Assume the claim is true. Using an idea due to A. M. Davie (see [**5**]), we show that $f_1 \in P(X)$ can be found satisfying (4) and (5).

Let $N=\{h\in C(X): \|h\|_X\leqq 1,\ \|h\|_F\leqq \eta\}$. We have to show that f is in the closure of $N\cap P(X)$ in the topology of uniform convergence on compact subsets of X°. By the separation theorem and Riesz representation theorem we need only show that if ν is a measure with compact support in X° satisfying $|\nu(p)|\leqq 1$ for all $p\in N\cap P(X)$ then $|\nu(f)|\leqq 1$. (The set of functions on supp ν in the uniform closure of functions in $N\cap P(X)$ is a closed convex subset of the space of all functions on supp ν.) N is the unit ball in $C(X)$ with respect to the norm

$$\|g\|_* = \max\{\|g\|_X, \eta^{-1}\|g\|_F\}$$

which is equivalent to sup norm on X since $\eta>0$. By the Hahn-Banach theorem we can extend the functional $p\to\nu(p)$ from $P(X)$ to $C(X)$ and represent it by a measure ν_1 on X satisfying $|\nu_1(g)|\leqq 1$ for all $g\in N$. This implies that

(6) $|\nu_1|(X\backslash\bar{F})+\eta|\nu_1|, (\bar{F})\leqq 1$.

Let $K\subset\partial X$ be compact and suppose $\mu(K)=0$. Then there exists $b\in P(X)$ such that $b=1$ on K and $|b|<1$ on $X\backslash K$ [**2**, Theorem 8.7, p. 47, and Theorem 12.7, p. 58]. Since ν_1 and ν agree on $P(X)$ we have $\nu_1(b^n)=\nu(b^n)$ for all n. By dominated convergence we get $\nu_1(K)=0$. Hence $\nu_2=(\nu_1)_{\partial X}\ll\mu$. Put $\nu_3=\nu_1-\nu_2$. Let $\{p_n\}$ denote the polynomials converging pointwise to f in X° and w^* to f^* in $L^\infty(\mu)$. We then get

$$0 = (\nu_1-\nu)p_n = \nu_2(p_n)+\nu_3(p_n)-\nu(p_n)\to\nu_2(f^*)+\nu_3(f)-\nu(f).$$

Hence $|\nu(f)|=|\nu_2(f^*)+\nu_3(f)|\leqq 1$ as desired. The last inequality follows from (6) since $|f^*|<1$ a.e. $d\mu$ on ∂X and $|f^*|\leqq\eta$ a.e. $d\mu$ on $\bar{F}\backslash X^\circ$

It remains to prove the above claim about $|f^*|$. If the claim was not true then we could find a Borel set $B\subset\bar{F}\backslash X^\circ$ and a number $t>0$ such that for some $\lambda=\lambda_n$ (in the sum defining μ) $\lambda(B)>0$ and $|f^*|\geqq\eta+t$ a.e. $d\lambda$ on B. Let $\phi: D\to W_n$ be a conformal map where D is the unit disc and W_n is the component of X° corresponding to λ. Let ϕ^* denote the boundary values which exist a.e. on ∂D w.r.t. linear measure by a theorem of Fatou [**1**, Theorem 13, p. 6]. The same theorem of Fatou combined with well-known properties about ϕ and ϕ^* (see for example [**2**, Lemma 4.3, p. 149]) implies that we can assert:

(7) $f^*(\phi^*(z))=\lim_{r\to 1} f(\phi(z))$ a.e. on ∂D;

(8) $B'=(\phi^*)^{-1}(B)$ has positive linear measure.

Since $|f(\phi^*)|>\eta+t$ a.e. on B' we could find $\{z_n\}\subset D$ such that $\phi(z_n)\to y\in B$ and $\liminf|f(\phi(z_n))|\geqq\eta+t$. We show that this is impossible. Let $\{w_n\}$ be any sequence in X° converging to $y\in\bar{F}\backslash X^\circ$. We wish to show that $\limsup|f(w_n)|\leqq\eta$. Since f is continuous in D we need only consider the case where $|y|=1$. Choose a polynomial p such that $1=p(y)=\|p\|_D$ and

$\|fp\|_K \leqq \|f\|_F$. Since (v) is assumed to hold we get

$$\limsup |f(w_n)| = \limsup |f(w_n)p(w_n)| \leqq \|fp\|_{K\cup F} \leqq \|f\|_F$$

and hence (iii) follows from (v).

Proof of Theorem 2. The proof will be split into two parts. We first prove that $\partial V \cap \partial D$ is accessible from V if $V \not\subset \hat{S}^{H(S)}$. We then show that if $\partial V \cap \partial D$ is accessible from V it follows that $V \cap \hat{S}^{U(S)} = \varnothing$. This will clearly complete the proof.

Let V be a bounded component of $C \backslash \bar{S}$, $z_0 \in V$ and assume $\|f\|_S < |f(z_0)|$ for some $f \in H(S)$. We can assume $\|f\|_S < 1$ and $u(z_0) > 1$ where $u = \operatorname{Re} f$ denotes the real part of f. We first prove that $\partial V \cap \partial D$ is accessible from $\bar{V}$. Note first that $\partial V \cap \partial D \neq \varnothing$ by the maximum principle for harmonic functions. We choose $z_1 \in V$ and a number $r_1 > 1$ such that $u(z_1) > r_1$ and $\{z \in V : u(z) \geqq r_1\} \cap \{z : |z| < 1 - 2^{-1}\} = \varnothing$. Let V_1 be the component of $\{z \in V : u(z) > r_1\}$ containing z_1. Again $\partial V_1 \cap \partial D = \varnothing$ by the maximum principle for harmonic functions. Choose $z_2 \in V_1$ and a number r_2 such that $u(z_2) > r_2$ and $\{z \in V_1 : u(z) > r_2\} \cap \{z : |z| \leqq 1 - 2^{-2}\} = \varnothing$. We now let V_2 be the component of $\{z \in V_1 : u(z) > r_2\}$ containing z_2 and proceed by induction. In this way we construct a sequence $\{z_n\}_{n=1}^{\infty}$ and connected sets $V_n \subset V$ such that $\{z_n, z_{n+1}\} \subset V_n$ and $\{z : |z| < 1 - 2^{-n}\} \cap V_n = \varnothing$ for $n = 1, 2, \cdots$. The polygonal arcs γ_n can now be constructed such that $\gamma_n \subset V_n$ for all n.

Let us now assume that V has the property that $\partial V \cap \partial D$ is accessible from V. Let $z_0 \in V$ be arbitrary. We can assume $\{z : |z - z_0| < \delta\} \cap \gamma_n = \varnothing$ for some $\delta > 0$ and all n. (If necessary we could redefine a finite number of the γ_n's.)

Since V is connected we can find a rational function R_1 with poles only at z_1 such that $\|R_1\|_S < 1$ and $|R_1(z_0)| > 2$. Let O_n be open and connected and assume $\gamma_n \subset O_n \subset V$ and $z_0 \notin O_n$ for $n = 1, 2, \cdots$. By induction we define rational functions R_k with poles only at z_k such that $\|R_{k+1} - R_k\|_{C \backslash O_k} < 2^{-k-1}$ for $k = 1, 2, \cdots$. Assume $O_n \cap K \neq \varnothing$ only for finitely many n if K is a compact subset of D. It follows that $\{R_k\}$ converges uniformly on $K \cup S$ for each compact subset K of D to a function $g \in U(S)$ satisfying $|g(z_0)| > \|g\|_S$. The proof is now complete.

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