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REPRESENTATIONS OF LOCALLY CONVEX *-ALGEBRAS

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ABSTRACT. Conditions for a functional to be admissible on a locally convex *-algebra are defined. Let F be an admissible positive Hermitian functional on a commutative locally convex *-algebra; then it is shown that there exists a representation of A into a Hilbert space. Sufficient conditions for a functional F to be representable are also given.

1. By a locally convex algebra A we shall mean an algebra A, over the complex numbers C, which has associated with it a Hausdorff topology τ such that multiplication is separately continuous. A will be called a locally convex *-algebra if A has a continuous involution. If x is an element of A such that $x^*=x$ then x will be called Hermitian.

An element x of A is said to be bounded if for some nonzero complex number λ, the set $\{(\lambda x)^n : n \in N\}$ is bounded. The set of bounded elements of A will be denoted by A_0. Let B_1 denote the collection of all closed, convex, circled sets B that are also bounded and idempotent. If $B \in B_1$, then $A(B)$ will denote the subalgebra of A generated by B, i.e., $A(B)=\{\lambda x : \lambda \in C, x \in B\}$, and the equation

$$\|x\|_B = \inf\{\lambda > 0 : x \in \lambda B\}$$

defines a norm which makes $A(B)$ a normed algebra. A will be called pseudo-complete if each $A(B)$ is a Banach algebra. For each $x \in A$, the radius of boundedness of x, $\beta(x)$, is defined by $\beta(x)=\inf\{\lambda>0 : \{(x/\lambda)^n : n \in N\}$ is bounded$\}$ with $\infty=\inf\varnothing$. (For properties of β see **[1]**.)

Let A be a locally convex *-algebra, and let F be a linear functional on A. If $F(x^*)=(F(x))^-$ for all x in A, F will be called Hermitian. If $F(x^*x)\geqq 0$ for all x in A, then F will be called a positive functional.

2. **Admissible functionals.** Before defining admissible functionals consider the following:

LEMMA 1. *Let A be a pseudo-complete locally convex *-algebra and let x_0 be any element of A such that $\beta(x_0)<1$. Then there exists an element*

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y_0 of A, such that $2y_0-y_0^2=x_0$. In addition if x_0 is Hermitian, y_0 will also be Hermitian.

PROOF. Consider the function f defined in terms of the binomial series as follows:

$$f(x) = -\sum_{n=1}^{\infty}\binom{1/2}{n}(-z)^n.$$

f is defined and $2f(z)-[f(z)]^2=z$ for all $|z|\leqq 1$. Now consider the vector valued function $\sum_{n=1}^{\infty}\binom{1/2}{n}(-x_0)^n$. We show that this series converges. Let $\varepsilon>0$. Since $\beta(x_0)<1$ there exists **[1]** a $B\in B_1$ such that $x_0\in A(B)$ and $\|x_0\|_B<1$. Since f converges for $|z|\leqq 1$ there exists an n_0 such that for $p, q>n_0$

$$\left\|\sum_{n=p}^{q-1}\binom{1/2}{n}(-x_0)^n\right\|_B<\varepsilon.$$

Since $A(B)$ is complete we have that the vector valued series converges to an element y_0 of $A(B)$ such that $2y_0-y_0^2=x_0$.

Using this lemma one can prove the following theorem.

THEOREM 2. *Let A be a pseudo-complete locally convex *-algebra and let F be any positive functional on A, then $|F(x^*hx)|\leqq\beta(h)F(x^*x)$ for x in A and h Hermitian.*

Let F be a positive functional on A and define $L_F=\{x\in A: F(y^*x)=0$ for all y in $A\}$. Then L_F is a left ideal and we define $X_F=A/L_F$. We denote $x+L_F$ by $\bar{x}$, i.e., $\bar{x}=x+L_F$.

DEFINITION. Positive functionals F which satisfy the following conditions will be called admissible:

(1) $\sup\{F(x^*a^*ax)/F(x^*x): x\in A\}<\infty$ for all $a\in A_0$, and

(2) for each $x\in A$ there is an $x_0\in A_0$ such that $\bar{x}=\bar{x}_0$.

The following two corollaries follow from Theorem 2.

COROLLARY 2.1. *If A is a pseudo-complete locally convex *-algebra such that $A=A_0$, then any positive functional is admissible.*

COROLLARY 2.2. *If A is a Banach *-algebra, all positive functionals are admissible.*

We now construct an example of an admissible functional on an algebra where $A\neq A_0$.

Let X be a locally compact Hausdorff space, and let A be the algebra of all continuous complex valued functions on X. Let A have the topology of uniform convergence on compact subsets of X. Consider the functional $F: A\to C$ given by $F(f)=f(x_0)$ where x_0 is a fixed element of X. Since A

is pseudo-complete, the first condition is satisfied. To show that the second condition of admissibility is satisfied let $f \in A$. Let $g(x)=f(x_0)$ for all $x \in X$. Then $g \in A_0$ and $\bar{g}=f+L_F$.

3. **Representations.** Let A be an algebra over the complex numbers and X a vector space over C. A representation of A is a homomorphism of A into $L^*(X)$, the algebra of all linear transformations of X into itself. Before proving the next theorem consider:

LEMMA 3. *Let A be a locally convex *-algebra and let F be an admissible positive functional on A. If a and b are elements of A, then $\overline{(a+b)_0}=(\bar{a}_0+\bar{b}_0)$.*

THEOREM 4. *Let F be an admissible positive Hermitian functional on the commutative locally convex *-algebra A. Then there exists a representation $a \to T_a$ of A on a Hilbert space H such that $(T_a)^*=T_{a^*}$ for all $a \in A_0$.*

PROOF. Since A is commutative L_F is a two-sided ideal and hence $X_F=A/L_F$ is an algebra. Let $\bar{x}=x+L_F$ and define a scalar product in X_F by $(\bar{x}, \bar{y})=F(y^*x)$, $x, y \in A$. The completion of X_F with respect to the inner product will be called H, and H is a Hilbert space.

Let $\bar{x}_0$ be a fixed element of X_F (since F is admissible we may assume that $x_0 \in A_0$). Let $\bar{z} \in H$ and assume that $\bar{z}_n \to \bar{z}$ with $\bar{z}_n \in X_F$. Then

$$\|\bar{x}_0\bar{z}_n - \bar{x}_0\bar{z}_m\|^2 = F((z_n - z_m)^*x_0^*x_0(z_n - z_m)) \leqq M \|\bar{z}_n - \bar{z}_m\|^2,$$

where $M>0$, since F is admissible. Thus $\{\bar{x}_0\bar{z}_n\}$ is a Cauchy sequence with respect to the inner product norm, and hence the sequence converges to an element $\bar{y}$ of H. Similarly, we can show that if $\bar{w}_n \to \bar{z}$ with respect to the inner product norm, then $\{\bar{x}_0\bar{w}_n\}$ converges to $\bar{y}$. We thus define $\bar{x}_0\bar{z}=\bar{y}$.

We now define the mapping $a \to T_a$ of A into H by

$$T_a\bar{x} = \bar{a}_0\bar{x}, \qquad x \in H,$$

where $\bar{a}_0=\bar{a}$. Then $T_a \in L^*(H)$ and this relationship defines a representation.

Consider the restriction of the representation to A_0. Let $a \in A_0$. Since F is admissible we have

$$\|T_a(\bar{x})\|^2 = F(x^*a^*ax) \leqq M\|\bar{x}\|^2, \qquad \bar{x} \in X_F,$$

for some $M>0$. Hence T_a is a continuous function on X_F and thus T_a can be uniquely extended to a continuous function $\hat{T}_a$ on H. However if $\bar{x} \in H - X_F$, let $\{\bar{x}_n\}_{n=1}^{\infty}$ be a subset of X_F such that $\bar{x}_n \to \bar{x}$. Then

$$\hat{T}_a(\bar{x}) = \lim \hat{T}_a(\bar{x}_n) = \lim T_a(x_n) = \lim a\bar{x}_n = a\bar{x} = T_a(x),$$

by the definition of multiplication of elements of H by elements of X_F. Thus $\hat{T}_a = T_a$ and T_a is a continuous function on H for $a \in A_0$.

Since T_a is continuous, we can show that $(T_a)^* = T_{a^*}$ by showing that $(T_a)^*(\bar{x}) = T_{a^*}(\bar{x})$ for all $\bar{x} \in X_F$. Let $\bar{x}$ and $\bar{y}$ be elements of X_F, then

$$(T_a\bar{x},\bar{y}) = F(y^*ax) = (\bar{x},(\overline{a^*})\bar{y}) = (\bar{x},T_{a^*}\bar{y}).$$

Thus for $a \in A$ we have $T_a^* = T_{a^*}$.

DEFINITION. A representation $a \rightarrow T_a$ of A on X is called a *-representation provided $(T_a)^*$ exists and is equal to T_{a^*} for every $a \in A$.

COROLLARY 4.1. *If A_0 is also an algebra (e.g., if the product of bounded sets of A is bounded) then the restriction of the above representation to A_0 is a *-representation of A_0 on H.*

Let X be a vector space over the complex numbers. Let K be a subalgebra of the algebra of all linear operators on the linear space X. Let z be a fixed vector in X and let $X_z = \{T(z): T \in K\}$. Then X_z is an invariant subspace of X with respect to K. If there is an element z of a normed space X such that $X_z = \bar{X}$, then K is said to be topologically cyclic and z is called a topologically cyclic vector. A representation $x \rightarrow T_x$ of A on X is said to be topologically cyclic if, when $K = \{T_x : x \in A\}$, there is a z in X such that $\bar{X}_z = X$.

With these definitions we state the following corollary to Theorem 4.

COROLLARY 4.2. *Let A be a commutative locally convex *-algebra with identity. Let F be an admissible positive Hermitian functional on A; then the representation obtained above is topologically cyclic with a cyclic vector h_0 such that $F(x) = (T_x h_0, h_0)$, $x \in A$.*

PROOF. Let $h_0 = \bar{1} = 1 + L_F$. Then by definition $T_x h_0 = \bar{x}_0$, so that the set $\{T_x h_0 : x \in A\} = X_F$ and hence is dense in H. Thus h_0 is a topologically cyclic vector. Now let $x \in A$, then there exists $x_0 \in A$ such that $\bar{x} = \bar{x}_0$. Thus

$$F(1^*(x - x_0)) = F(x - x_0) = 0 \quad \text{or} \quad F(x) = F(x_0).$$

Therefore $(T_x h_0, h_0) = (\bar{x} h_0, h_0) = F(x_0) = F(x)$ for all $x \in A$.

4. **Representable functionals.** Let F be functional on the locally convex *-algebra A and let $a \rightarrow T_a$ be a representation of A on a Hilbert space H such that the restriction of the representation to A_0 is a *-representation of A_0 on H. Then F is said to be represented by $a \rightarrow T_a$ provided there exists a topologically cyclic vector $h_0 \in H$ such that $F(x) = (T_x h_0, h_0)$ for all $x \in A$.

DEFINITION. Let $x \to T_x$ be a representation of A on H. Let $M = \{h \in H : T_x h = 0 \text{ for all } x \in A\}$. If $M = \{0\}$, we say that the representation is essential.

The following lemma is found in Rickart [**4**].

LEMMA 5. *If the representation $x \to T_x$ is essential, then each of the subspaces $H_h = \{T_x h : x \in A\}$ is cyclic with h as a cyclic vector.*

THEOREM 6. *Let F be a Hermitian functional on the pseudo-complete commutative locally convex *-algebra A. Then in order for F to be representable, it is sufficient that*

(1) *for each $x \in A$, there is an x_0 in A_0 such that $\bar{x} = \bar{x}_0$, and*

(2) $|F(x)|^2 \leqq \mu F(x^* x)$, $x \in A$,

where μ is a positive real constant independent of X.

PROOF. Assume that F satisfies the conditions and denote by A_1 the pseudo-complete locally convex *-algebra obtained by adjoining the identity element to A. Extend the functional F to A_1 by the definition, $F(x+\alpha) = F(x) + \mu\alpha$ for $x \in A$ and α a scalar. Then F is a positive functional on A_1 and Theorem 2 guarantees that the first condition of admissibility is satisfied on A_1. To show that the second condition is satisfied, let $x + \alpha \in A_1$. Then by hypothesis there exists $x_0 \in A_0$ such that $\bar{x}_0 = \bar{x}$. Consider $x_0 + \alpha$. We show that $\overline{(x_0+\alpha)} = \overline{(x+\alpha)_0}$.

$$\begin{aligned} |F[(y+\beta)^*((x_0+\alpha)-(x+\alpha))]|^2 &= |F[(y+\beta)^*(x_0-x)]|^2 \\ &= |F(y^*(x-x_0)) + F(\bar{\beta}(x_0-x))|^2 \\ &= |0 + \bar{\beta}F(x_0-x)|^2 \\ &\leqq |\beta|^2 F[(x_0-x)^*(x_0-x)] = 0 \end{aligned}$$

since $\bar{x}_0 = \bar{x}$, and $(x - x_0) \in A$.

Hence by Corollary 4.2 there exists a representation $x \to T_x$ of A_1 on H defined by $T_{(a+\alpha)}\bar{x} = \overline{(a+\alpha)_0 x}$ and such that $F(a+\alpha) = (T_{a+\alpha}h_0, h_0)$ for some $h_0 \in H$. Now let

$$N = \{h \in H : T_a h = \theta \text{ for all } a \in A\}.$$

Consider the restriction of $a \to T_a$ to the space $N^\perp$, where

$$N^\perp = \{h \in H : (h, n) = 0 \text{ for all } n \in N\}.$$

The restriction of the representation is essential.

Let $h_0 = h_0' + h_0''$ where $h_0' \in N^\perp$ and $h_0'' \in N$. Then for all $a \in A$ we have that

$$\begin{aligned} F(a) &= (T_a h_0, h_0) = (T_a h_0', h_0) = (h_0', T_{a0}^*(h_0' + h_0'')) \\ &= (h_0', T_{a0}^* h_0') = (T_a h_0', h_0'). \end{aligned}$$

Thus if we let $H_0 = \{T_a h_0' : a \in A\}$ and apply Lemma 5 we have that F is representable.

COROLLARY 6.1. *If A has an identity element then every positive functional which implies condition* (1) *is representable.*

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