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PROVING KNESER'S THEOREM FOR FINITE GROUPS BY ANOTHER e-TRANSFORM*

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ABSTRACT. Although neither the result nor the e-transformation is new, a new order for the successive transformations is prescribed. From this follow some interesting properties which in turn imply the result.

Let A, B and C be subsets of a finite abelian group G, with a, b and c the respective elements. Suppose further that $A+B=C$.

DEFINITION. Let $\bar{A}=\{x:x+B\subset C\}$. We say A is full whenever $A=\bar{A}$. Notice that $A\subset\bar{A}$ and $\bar{A}+B=C$.

DEFINITION. Let $\Delta C=\{x:x+C\subset C\}$. It is obvious that $0\in\Delta C$, $C+\Delta C\subset C$, $\Delta A\subset\Delta C$ (in fact $\Delta A=\Delta C$ if A is full). Also $C+S\subset C$ implies that $S\subset\Delta C$. Hence $\Delta C+\Delta C=\Delta C$, so that ΔC is a subgroup of G. We now use the usual Dyson e-transform. H. B. Mann's e-transform would serve equally well with only the obvious modifications in the proof, and indeed the proof was originally done using that transform.

For e in A, a full set, and $0\in B$ another full set,

$$A\to A'=A\cup(B+e),$$
$$B\to B'=B\cap(A-e),$$
$$C\to C'=A'+B'.$$

Also let

$$A^*=\{a:a\in A',\ a\notin A\},$$
$$B^*=\{b:b\in B,\ b\notin B'\}.$$

We have the following usual properties:

P_0. $C'\subset C$,

P_1. $B^*+e=A^*$,

P_2. $|A|+|B|=|A'|+|B'|$,

P_3. $0\in B'$.

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For any b' such that $b'+A'\subset C'$, we clearly have $b'+A\subset C$ and $b'+(B+e)\subset C$, which implies, since A and B are full, that b' is in both B and $A-e$. Thus b' is in B' showing that

P_4. B' is full.

Let $H=C'$. Since B' is full, $H=B'$; so that $B'+H=B'\subset A^*-e$. This implies

P_5. $e+B'+H\subset A$.

THEOREM. $|A|+|B|\leqq|C|+|\Delta C|$.

LEMMA 1. *For any a in A such that $a+B+H\not\subset C$*

$$|A\cap(a+H)|+|C'|\leqq|C|.$$

PROOF. By hypothesis, there is an element b of B so that $a+b+H\not\subset C$. Since $C'+H\subset C'$ and $C'\subset C$, C' must be disjoint from the coset $a+b+H$. The set $[A\cap(a+H)]+b$ is contained in this coset and also in C. Hence C has at least $|[A\cap(a+H)]+b|=|A\cap(a+H)|$ more elements than C' has.

We may inductively assume the theorem to be true for any A_1, B_1 and C_1 such that $A_1+B_1=C_1$ and $|B_1|<|B|$. Assuming for now that e can be chosen so that $|B'|<|B|$ we clearly have $(A'+H)+B'=C'$; so by the inductive assumption:

$$|A'+H|+|B'|\leqq|C'|+|H|. \tag{1}$$

LEMMA 2. *For any $a\in A$ such that $a+H\not\subset A$,*

$$|A|+|B|\leqq|C|+|A^*\cap(a+H)|.$$

PROOF. If $a+B+H$ were contained in C, then $a+H$ would be contained in A, since A is full. This is, however, not the case so it follows that $a+B+H\not\subset C$. The inequality of Lemma 1 thus holds and adding this to inequality (1) and the following three easy relations yield the lemma after massive cancellation:

$$\begin{aligned}|A|+|B|&=|A'|+|B'|,\\ |A'\cap(a+H)|&=|A\cap(a+H)|+|A^*\cap(a+H)|,\\ |H|+|A'|&\leqq|A'\cap(a+H)|+|A'+H|.\end{aligned}$$

Let the images of A, B and C under the transformation by $d\in A$ be the sets A'_d, B'_d, C'_d respectively, with difference sets A^*_d and B^*_d.

Unless $A+B\subset A$, there must be a d in A with $d+B\not\subset A$. Thus B^*_d is not empty. Choose e in A so that B^*_e is not empty, but minimal in the sense that no nonempty B^*_d is properly contained in it. This means that if $B^*_d\subset B^*_e$ then either $B^*_d=B^*_e$ or $B^*_d=\varnothing$. As before, we call A'_e, B'_e, A^*_e, and B^*_e respectively A', B', A^* and B^*.

By P_3 and P_5, $e+H\subset A$ so we can transform by any $e+h$, where $h\in H$. Let

$$K=\{h:h\in H,\ B^*_{e+h}=\varnothing\}.$$

LEMMA 3. *For any a^* in A^*, $(a^*+H)\cap A=a^*+K$.*

PROOF. By P_5, $e+H+B'\subset A$, so that no b' in B' could even be removed by a transformation under $e+h$. Hence, $B^*_{e+h}\subset B^*$. Thus by the minimal choice of e, we have either $B^*_{e+h}=\varnothing$ or $B^*_{e+h}=B^*$. If h is in K then $B^*_{e+h}=\varnothing$, so that $A^*_{e+h}=\varnothing$ and $a^*+h\in B+e+h\subset A'_{e+h}=A$. If h is not in K, $B^*_{e+h}=B^*$, so that by P_1, $A^*_{e+h}=e+h+B^*_{e+h}=e+h+B^*=h+A^*$. Therefore a^*+h is in A^*_{e+h}, a set which is disjoint from A. This shows that a^*+h is in A if and only if h is in K, proving the lemma.

LEMMA 4. *For any a in A, $|(a+H)\cap A^*|\leqq|\Delta K|$.*

This is trivial if $(a+H)\cap A^*$ is empty, so assume it is nonempty. For any a_1^* and a_2^* in $(a+H)\cap A^*$, Lemma 3 implies $(a+H)\cap A=a_1^*+K=a_2^*+K$. Thus $K+(a_1^*-a_2^*)\subset K$, so that

$$((a+H)\cap A)-((a+H)\cap A^*)\subset\Delta K.$$

Moreover,

$$\begin{aligned}|(a+H)\cap A^*| &= |((a+H)\cap A^*)-a_1^*|\\ &\leqq |((a+H)\cap A^*)-((a+H)\cap A^*)|\leqq|\Delta K|,\end{aligned}$$

which proves the lemma.

PROOF OF THE THEOREM. If B is empty, so is C; making $\Delta C=G$, and the theorem trivially follows. We may thus assume B is nonempty and moreover that 0 is an element of B by taking translates of B and C if necessary. We may also assume that A and B are full by replacing A by $\bar{A}$ and then B by $\bar{B}$, since this only increases the left-hand side of the inequality.

If $B+A\subset A$, then $B\subset\Delta A=\Delta C$. Since $0\in B$, we have $A\subset C$ and hence $|A|+|B|\leqq|C|+|\Delta C|$.

If $B+A\not\subset A$, e may be chosen so as to make B^* minimal as before.

The proof now divides into three cases.

Case I. For every a in A, $a+H\subset A$.

In this case $A+H\subset A$ so that $H\subset\Delta A=\Delta C$. Hence

$$|A|+|B|=|A'|+|B'|\leqq|C'|+|H|\leqq|C|+|\Delta C|$$

follows from P_2, (1), P_0 and the last containment.

Case II. There is an a in A with $a+H\not\subset A$ and $(a+H)\cap A^*$ is empty.

In this case the theorem follows immediately from Lemma 2.

Case III. For every a in A, either $a+H\subset A$ or $(a+H)\cap A^*$ is nonempty. Moreover, for some a in A, $a+H\not\subset A$.

In this case we can show $A+\Delta K\subset A$. For any a in A, if $a+H\subset A$, then K is contained in the subgroup H, hence so is ΔK and thus $a+\Delta K\subset A$. If $a+H\not\subset A$, then there is an a^* in $A^*\cap(a+H)$. By Lemma 3,

$$a + \Delta K \subset (a + H) \cap A + \Delta K = a^* + K + \Delta K = a^* + K \subset A,$$

which proves the assertion.

This implies that $\Delta K\subset\Delta A=\Delta C$, so that by Lemma 4

$$|(a + H) \cap A^*| \leqq |\Delta C|. \tag{2}$$

There is some a in A with $a+H\not\subset A$, so that Lemma 2 together with (2) now imply the theorem.

This statement of Kneser's theorem suggests the following conjectures for sets of nonnegative integers.

Let $A+B=C$. Let $H^{(n)}=\{x:c\in C,\ x+c\leqq n \text{ implies } x+c\in C\}$; then

$$\min_{m\leqq n}\frac{C(m)+H^{(n)}(m)}{m+1}\geqq\min_{m\leqq n}\frac{A(m)+B(m)}{m+1}.$$

Furthermore, it seems that $H^{(n)}$ may be replaced by

$$J^{(n)} = \{x:c \in C,\ c \leqq n \text{ implies } x + c \in C\}.$$

Either of these conjectures implies both Mann's theorem and Kneser's theorem for sets of integers.

Bibliography

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