THE APPROXIMATION OF ONE-ONE MEASURABLE TRANSFORMATIONS BY MEASURE PRESERVING HOMEOMORPHISMS

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ABSTRACT. This paper contains two results related to the material in [2]. Suppose $f$ is a one-one transformation of the open unit interval $I^n$ (where $n \geq 2$) onto $I^n$. 1. If $f$ is absolutely measurable and $\varepsilon > 0$, then there is an absolutely measurable homeomorphism $\varphi_\varepsilon$ of $I^n$ onto $I^n$ such that $m(\{x: f(x) \neq \varphi_\varepsilon(x) \text{ or } f^{-1}(x) \neq \varphi_\varepsilon^{-1}(x)\}) < \varepsilon$, where $m$ denotes $n$-dimensional Lebesgue measure. 2. Suppose $\mu$ is either (1) a nonatomic, finite Borel measure on $I^n$ such that $\mu(G) > 0$ for every nonempty open subset $G$ of $I^n$, or (2) the completion of a measure of type (1). If $f$ is $\mu$-measure preserving and $\varepsilon > 0$, then there is a $\mu$-measure preserving homeomorphism $\varphi_\varepsilon$ of $I^n$ onto $I^n$ such that $\mu(\{x: f(x) \neq \varphi_\varepsilon(x)\}) < \varepsilon$.

1. For any subset $S$ of $n$-dimensional Euclidean space $R^n$, denote by $\mathcal{M}(S)$ the set of all measures $\mu$ such that $\mu$ is either (1) a nonatomic, finite, Borel measure on $S$ such that $\mu(G) > 0$ for every nonempty open subset $G$ of $S$ or (2) the completion of a measure of type (1). If, for $i=1, 2$, $S_i$ is a subset of $R^n$ and $\mu_i \in \mathcal{M}(S_i)$, and $f$ is a one-one transformation of $S_1$ onto $S_2$, then we say that $f$ carries $\mu_1$ into $\mu_2$ provided $f[D(\mu_1)] = D(\mu_2)$ and $\mu_2(f[A]) = \mu_1(A)$ for every $A$ in $D(\mu_1)$, where $D(\mu_i)$ is the domain of $\mu_i$. If $S_1 = S_2$ and $\mu_1 = \mu_2$, then we say that $f$ is $\mu_1$-measure preserving.

In this note, we show how a minor modification of the proof of Theorem 5 of [2] yields the following result.

THEOREM 1. Suppose $\mu_1, \mu_2 \in \mathcal{M}(I^n)$, where $n \geq 2$ and $I^n$ denotes the open unit interval in $R^n$, and $f$ is a one-one transformation of $I^n$ onto $I^n$ which carries $\mu_1$ into $\mu_2$. For every $\varepsilon > 0$, there is a homeomorphism $\varphi_\varepsilon$ of $I^n$ onto $I^n$ which carries $\mu_1$ into $\mu_2$ such that

$$\mu_1(\{x: f(x) \neq \varphi_\varepsilon(x)\}) = \mu_2(\{x: f^{-1}(x) \neq \varphi_\varepsilon^{-1}(x)\}) < \varepsilon.$$
REMARKS. The author has been informed recently by J. C. Oxtoby that he has, in his paper *Approximation by measure-preserving homeomorphisms*, generalized Theorem 1 (with \( \mu_1 = \mu_2 \)). In doing so, he re-proved this statement. His work was done independently and was done after the work in this paper.

A one-one transformation \( f \) of \( I^n \) onto \( I^n \) is called absolutely measurable [2] if \( f[A] \) and \( f^{-1}[A] \) are Lebesgue measurable for every Lebesgue measurable subset \( A \) of \( I^n \).

We then obtain the following result as a corollary to Theorem 1.

**THEOREM 2.** If \( f \) is an absolutely measurable, one-one transformation of \( I^n \) onto \( I^n \) (where \( n \geq 2 \)) and \( \epsilon > 0 \), then there is an absolutely measurable homeomorphism \( \varphi_\epsilon \) of \( I^n \) onto \( I^n \) such that

\[
m(\{x : f(x) \neq \varphi_\epsilon(x) \text{ or } f^{-1}(x) \neq \varphi_\epsilon^{-1}(x)\}) < \epsilon,
\]

where \( m \) denotes \( n \)-dimensional Lebesgue measure.

2. In this section \( n \) will always denote a fixed integer \( \geq 2 \). By an \((n-1)\)-dimensional interval in \( R^n \) we mean a set of the form

\[
\{(x_1, \cdots, x_n) \in R^n : x_k = c \} \cap \prod \{[a_j, b_j] : j = 1, \cdots, n \},
\]

where \( k \) is an integer such that \( 1 \leq k \leq n \), \( c \) is a real number, and, for \( j = 1, \cdots, n \), \( a_j \) and \( b_j \) are real numbers such that \( a_j < b_j \). For any subset \( A \) of \( R^n \), we denote the interior of \( A \), the closure of \( A \), and the boundary of \( A \) by \( \text{int} \ A \), \( \text{cl} \ A \), and \( \text{bdry} \ A \), respectively.

**DEFINITION.** A subset \( P \) of \( R^n \) is called a \( p \)-set if \( P \) is a combinatorial \( n \)-ball (see p. 18 of [1]) and \( \text{bdry} \ P \) is the union of a finite number of \((n-1)\)-dimensional intervals.

**REMARKS.** (1) The \( p \)-sets used in the proof of Theorem 1 (and Theorem 5 of [2]) can be chosen to be very simple "snake-like" objects.

(2) The author wishes to thank Dr. L. C. Glaser for answering a number of questions concerning Lemma 5 of [2].

The following statement follows from Corollary 3 of [3] and Lemma 5 of [2].

**LEMMA 1.** Suppose, for \( i = 1, 2 \), that \( \{P(i,j) : j = 1, \cdots, r \} \) is a disjoint family of \( p \)-sets contained in the interior of the \( p \)-set \( P(i) \). For \( i = 1, 2 \), let \( Q(i) = P(i) \sim \bigcup \{\text{int} \ P(i,j) : j = 1, \cdots, r \} \), and suppose \( \mu_i \in \mathcal{M}(Q(i)) \) and \( \mu_i(\text{bdry} \ Q(i)) = 0 \). If \( \mu_1(Q(1)) = \mu_2(Q(2)) \), then every homeomorphism \( \varphi \) of \( \text{bdry} \ P(1) \) onto \( \text{bdry} \ P(2) \) can be extended to a homeomorphism \( \varphi^* \) of \( Q(1) \) onto \( Q(2) \) which carries \( \mu_1 \) into \( \mu_2 \) such that \( \varphi^*[\text{bdry} \ P(1,j)] = \text{bdry} \ P(2,j) \) for \( j = 1, \cdots, r \).
The following statement follows easily from the definition of sectionally zero dimensional set [2, p. 263].

**Lemma 2.** Suppose \( K \) is a sectionally zero dimensional, compact set contained in the interior of the \( P \)-set \( P \) such that \( m(K) < \gamma < m(P) \). Then there is a \( P \)-set \( Q \) such that \( K \subseteq \text{int} \ Q, \ Q \subseteq \text{int} \ P, \) and \( m(Q) = \gamma \).

**Lemma 3.** Suppose \( P, Q \) are \( P \)-sets contained in \( I^n \), and \( S \) and \( T \) are compact, sectionally zero dimensional sets contained in \( \text{int} \ P \) and \( \text{int} \ Q \), respectively. If \( \phi \) is an \( m \)-measure preserving homeomorphism of \( S \) onto \( T \) and \( m(P) = m(Q) \), then \( \phi \) can be extended to an \( m \)-measure preserving homeomorphism of \( P \) onto \( Q \).

We obtain a proof of Lemma 3 by making the following modifications in the proof of Theorem 1 of [2]. At the \( k \)th step of the definition of the auxiliary sets, since \( m(S_{i_1, \ldots, i_k}) = m(T_{i_1, \ldots, i_k}) \) for \( j_1 \leq m_1, \ldots, j_k \leq m_{j_1, \ldots, j_{k-1}} \), by Lemma 2, the \( P \)-sets \( P_{i_1, \ldots, i_k}, Q_{i_1, \ldots, i_k} \) can be chosen so that \( m(P_{i_1, \ldots, i_k}) = m(Q_{i_1, \ldots, i_k}) \) for \( j_1 \leq m_1, \ldots, j_k \leq m_{j_1, \ldots, j_{k-1}} \). Then, at the \( k \)th step in defining the extension of \( \phi \), instead of Lemma 5 of [2], we use Lemma 1.

**Remark.** In proving Theorem 1 of [2], C. Goffman uses Lemma 4 of [2]. Lemma 4 of [2] is false. However, if the following sentence is added to the hypothesis of Lemma 4, then the resulting lemma is true. For each \( I_i \), there is an interval \( J_i \) such that \( F_i \subseteq \text{int} \ J_i \) and \( J_i \subseteq P \). The modified version of Lemma 4 of [2] is sufficient for the proof of Lemma 3 (and Theorem 1 of [2]).

**Proof of Theorem 1.** If \( \mu_1 = \mu_2 = m \), Theorem 1 follows from Lemma 3 in exactly the same way as Theorem 5 of [2] follows from Theorem 1 of [2]. Now, suppose \( \mu_1, \mu_2 \) are arbitrary elements of \( \mathcal{M}(I^n) \) and \( f \) is as hypothesized. First, note that either both \( \mu_1 \) and \( \mu_2 \) are of type (1) or both \( \mu_1 \) and \( \mu_2 \) are of type (2). Hence, we may assume that both \( \mu_1 \) and \( \mu_2 \) are of type (2) and that \( \mu_1(I^n) = 1 \). By Theorem 2 of [3], there are homeomorphisms \( \psi \) and \( \varphi \) of \( \text{cl} \ I^n \) onto \( \text{cl} \ I^n \) such that \( \psi \) carries \( m \) into \( \mu_1 \) and \( \varphi \) carries \( \mu_2 \) to \( m \). Then \( f^* = \varphi \circ f \circ \psi \) is \( m \)-measure preserving. If \( \theta \) is an \( m \)-measure preserving homeomorphism of \( I^n \) onto \( I^n \) such that \( m(\{x : f^*(x) \neq \theta(x)\}) < \varepsilon \), then \( \varphi_\varepsilon = \varphi^{-1} \circ \theta \circ \psi^{-1} \) is the required homeomorphism.

**Proof of Theorem 2.** Suppose \( f \) is as hypothesized. For any Lebesgue measurable subset \( A \) of \( I^n \), let \( \mu(A) = m(f^{-1}[A]) \). Then \( \mu \in \mathcal{M}(I^n) \) and \( f \) carries \( m \) into \( \mu \). Let \( \delta > 0 \) be such that \( \delta \leq \varepsilon \) and, if \( m(A) < \delta \), then \( \mu(A) < \varepsilon \). By Theorem 1, there is a homeomorphism \( \varphi_\varepsilon \) of \( I^n \) onto \( I^n \) carrying \( m \) into \( \mu \) such that \( m(\{x : f(x) \neq \varphi_\varepsilon(x)\}) < \delta \). It is clear that \( \varphi_\varepsilon \) is the required homeomorphism.
Remarks. In proving Theorem 1 with \( \mu_1 = \mu_2 \), J. C. Oxtoby showed that \( \varphi_\varepsilon \) could be chosen to be a homeomorphism of \( \text{cl} \ I^n \) onto \( \text{cl} \ I^n \) such that \( \varphi_\varepsilon \) is equal to the identity outside of some closed interval contained in \( I^n \). It is clear that (a) the proof of Theorem 1 given here yields this, too, and (b) the \( \varphi_\varepsilon \) in Theorem 2 may be chosen to have these properties. Furthermore, in Theorem 1, \( \varphi_\varepsilon \) can be chosen to be a homeomorphism of \( \text{cl} \ I^n \) onto \( \text{cl} \ I^n \) which is equal to the identity on \( \text{bdry} \ I^n \).

References


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