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THE APPROXIMATION OF ONE-ONE MEASURABLE TRANSFORMATIONS BY MEASURE PRESERVING HOMEOMORPHISMS

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ABSTRACT. This paper contains two results related to the material in [2]. Suppose f is a one-one transformation of the open unit interval I^n (where $n\geqq 2$) onto I^n. 1. If f is absolutely measureable and $\varepsilon>0$, then there is an absolutely measurable homeomorphism φ_ε of I^n onto I^n such that $m(\{x:f(x)\neq\varphi_\varepsilon(x)$ or $f^{-1}(x)\neq \varphi_\varepsilon^{-1}(x)\})<\varepsilon$, where m denotes n-dimensional Lebesgue measure. 2. Suppose μ is either (1) a nonatomic, finite Borel measure on I^n such that $\mu(G)>0$ for every nonempty open subset G of I^n, or (2) the completion of a measure of type (1). If f is μ-measure preserving and $\varepsilon>0$, then there is a μ-measure preserving homeomorphism φ_ε of I^n onto I^n such that $\mu(\{x:f(x)\neq\varphi_\varepsilon(x)\})<\varepsilon$.

1. For any subset S of n-dimensional Euclidean space R^n, denote by $\mathscr{M}(S)$ the set of all measures μ such that μ is either (1) a nonatomic, finite, Borel measure on S such that $\mu(G)>0$ for every nonempty open subset G of S or (2) the completion of a measure of type (1). If, for $i=1, 2$, S_i is a subset of R^n and $\mu_i\in\mathscr{M}(S_i)$, and f is a one-one transformation of S_1 onto S_2, then we say that f carries μ_1 into μ_2 provided $f[D(\mu_1)]=D(\mu_2)$ and $\mu_2(f[A])=\mu_1(A)$ for every A in $D(\mu_1)$, where $D(\mu_i)$ is the domain of μ_i. If $S_1=S_2$ and $\mu_1=\mu_2$, then we say that f is μ_1-measure preserving.

In this note, we show how a minor modification of the proof of Theorem 5 of [2] yields the following result.

THEOREM 1. *Suppose $\mu_1, \mu_2\in\mathscr{M}(I^n)$, where $n\geqq 2$ and I^n denotes the open unit interval in R^n, and f is a one-one transformation of I^n onto I^n which carries μ_1 into μ_2. For every $\varepsilon>0$, there is a homeomorphism φ_ε of I^n onto I^n which carries μ_1 into μ_2 such that*

$$\mu_1(\{x: f(x)\neq\varphi_\varepsilon(x)\})=\mu_2(\{x: f^{-1}(x)\neq\varphi_\varepsilon^{-1}(x)\})<\varepsilon.$$

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REMARKS. The author has been informed recently by J. C. Oxtoby that he has, in his paper *Approximation by measure-preserving homeomorphisms*, generalized Theorem 1 (with $\mu_1=\mu_2$). In doing so, he re-proved this statement. His work was done independently and was done after the work in this paper.

A one-one transformation f of I^n onto I^n is called absolutely measurable [2] if $f[A]$ and $f^{-1}[A]$ are Lebesgue measurable for every Lebesgue measurable subset A of I^n.

We then obtain the following result as a corollary to Theorem 1.

THEOREM 2. *If f is an absolutely measurable, one-one transformation of I^n onto I^n (where $n \geqq 2$) and $\varepsilon > 0$, then there is an absolutely measurable homeomorphism φ_ε of I^n onto I^n such that*

$$m(\{x : f(x) \neq \varphi_\varepsilon(x) \text{ or } f^{-1}(x) \neq \varphi_\varepsilon^{-1}(x)\}) < \varepsilon,$$

where m denotes n-dimensional Lebesgue measure.

2. In this section n will always denote a fixed integer $\geqq 2$. By an $(n-1)$-dimensional interval in R^n we mean a set of the form

$$\{(x_1, \cdots, x_n) \in R^n : x_k = c\} \cap \prod \{[a_j, b_j] : j = 1, \cdots, n\},$$

where k is an integer such that $1 \leqq k \leqq n$, c is a real number, and, for $j=1, \cdots, n$, a_j and b_j are real numbers such that $a_j < b_j$. For any subset A of R^n, we denote the interior of A, the closure of A, and the boundary of A by int A, cl A, and bdry A, respectively.

DEFINITION. A subset P of R^n is called a p-set if P is a combinatorial n-ball (see p. 18 of [**1**]) and bdry P is the union of a finite number of $(n-1)$-dimensional intervals.

REMARKS. (1) The p-sets used in the proof of Theorem 1 (and Theorem 5 of [**2**]) can be chosen to be very simple "snake-like" objects.

(2) The author wishes to thank Dr. L. C. Glaser for answering a number of questions concerning Lemma 5 of [**2**].

The following statement follows from Corollary 3 of [**3**] and Lemma 5 of [**2**].

LEMMA 1. *Suppose, for $i=1, 2$, that $\{P(i,j) : j=1, \cdots, r\}$ is a disjoint family of p-sets contained in the interior of the p-set $P(i)$. For $i=1, 2$, let $Q(i)=P(i) \sim \bigcup \{\text{int}\, P(i,j) : j=1, \cdots, r\}$, and suppose $\mu_i \in \mathscr{M}(Q(i))$ and $\mu_i(\text{bdry}\, Q(i))=0$. If $\mu_1(Q(1))=\mu_2(Q(2))$, then every homeomorphism φ of* bdry $P(1)$ *onto* bdry $P(2)$ *can be extended to a homeomorphism φ^* of $Q(1)$ onto $Q(2)$ which carries μ_1 into μ_2 such that $\varphi^*[\text{bdry}\, P(1,j)]=$* bdry $P(2,j)$ *for $j=1, \cdots, r$.*

The following statement follows easily from the definition of sectionally zero dimensional set [**2**, p. 263].

LEMMA 2. *Suppose K is a sectionally zero dimensional, compact set contained in the interior of the p-set P such that $m(K)<\gamma<m(P)$. Then there is a p-set Q such that $K\subset \operatorname{int} Q$, $Q\subset \operatorname{int} P$, and $m(Q)=\gamma$.*

LEMMA 3. *Suppose P, Q are p-sets contained in I^n, and S and T are compact, sectionally zero dimensional sets contained in* $\operatorname{int} P$ *and* $\operatorname{int} Q$, *respectively. If φ is an m-measure preserving homeomorphism of S onto T and $m(P)=m(Q)$, then φ can be extended to an m-measure preserving homeomorphism of P onto Q.*

We obtain a proof of Lemma 3 by making the following modifications in the proof of Theorem 1 of [**2**]. At the kth step of the definition of the auxiliary sets, since $m(S_{j_1\dots j_k})=m(T_{j_1\dots j_k})$ for $j_1\leqq m_1,\cdots,j_k\leqq m_{j_1\dots j_{k-1}}$, by Lemma 2, the p-sets $P_{j_1\dots j_k}$, $Q_{j_1\dots j_k}$ can be chosen so that $m(P_{j_1\dots j_k})=m(Q_{j_1\dots j_k})$ for $j_1\leqq m_1,\cdots,j_k\leqq m_{j_1\dots j_{k-1}}$. Then, at the kth step in defining the extension of φ, instead of Lemma 5 of [**2**], we use Lemma 1.

REMARK. In proving Theorem 1 of [**2**], C. Goffman uses Lemma 4 of [**2**]. Lemma 4 of [**2**] is false. However, if the following sentence is added to the hypothesis of Lemma 4, then the resulting lemma is true. For each i, there is an interval J_i such that $F_i\subset \operatorname{int} J_i$ and $J_i\subset P$. The modified version of Lemma 4 of [**2**] is sufficient for the proof of Lemma 3 (and Theorem 1 of [**2**]).

PROOF OF THEOREM 1. If $\mu_1=\mu_2=m$, Theorem 1 follows from Lemma 3 in exactly the same way as Theorem 5 of [**2**] follows from Theorem 1 of [**2**]. Now, suppose μ_1, μ_2 are arbitrary elements of $\mathscr{M}(I^n)$ and f is as hypothesized. First, note that either both μ_1 and μ_2 are of type (1) or both μ_1 and μ_2 are of type (2). Hence, we may assume that both μ_1 and μ_2 are of type (2) and that $\mu_1(I^n)=1$. By Theorem 2 of [**3**], there are homeomorphisms ψ and φ of $\operatorname{cl} I^n$ onto $\operatorname{cl} I^n$ such that ψ carries m into μ_1 and φ carries μ_2 to m. Then $f^*=\varphi\circ f\circ\psi$ is m-measure preserving. If θ is an m-measure preserving homeomorphism of I^n onto I^n such that $m(\{x:f^*(x)\neq\theta(x)\})<\varepsilon$, then $\varphi_\varepsilon=\varphi^{-1}\circ\theta\circ\psi^{-1}$ is the required homeomorphism.

PROOF OF THEOREM 2. Suppose f is as hypothesized. For any Lebesgue measurable subset A of I^n, let $\mu(A)=m(f^{-1}[A])$. Then $\mu\in\mathscr{M}(I^n)$ and f carries m into μ. Let $\delta>0$ be such that $\delta\leqq\varepsilon$ and, if $m(A)<\delta$, then $\mu(A)<\varepsilon$. By Theorem 1, there is a homeomorphism φ_ε of I^n onto I^n carrying m into μ such that $m(\{x:f(x)\neq\varphi_\varepsilon(x)\})<\delta$. It is clear that φ_ε is the required homeomorphism.

REMARKS. In proving Theorem 1 with $\mu_1=\mu_2$, J. C. Oxtoby showed that φ_ε could be chosen to be a homeomorphism of cl I^n onto cl I^n such that φ_ε is equal to the identity outside of some closed interval contained in I^n. It is clear that (a) the proof of Theorem 1 given here yields this, too, and (b) the φ_ε in Theorem 2 may be chosen to have these properties. Furthermore, in Theorem 1, φ_ε can be chosen to be a homeomorphism of cl I^n onto cl I^n which is equal to the identity on bdry I^n.

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