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A NOTE ON SPACES WITH NORMAL PRODUCT WITH SOME COMPACT SPACE

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ABSTRACT. For compact X with $\log|X| \geqq \aleph$, $X \times Z$ is normal only if Z is $\aleph$-collectionwise normal. If Z is also semimetric or $\aleph$-metacompact, it is then $\aleph$-paracompact.

All spaces herein are to be Hausdorff. The density function is denoted $d(\ )$.

LEMMA 1. *If Y is a nondegenerate AE for X, and Y^X has the compact-open topology*, $d(Y^X) \geqq d(Y) \cdot \log|X|$.

PROOF. $\prod Y_x$, $x \in X$, has density $d(Y) \cdot \log|X|$ [**3**, Theorem 4.5]. Y^X with the p-topology is a dense subspace of $\prod Y_x$, so its density is no less. The result follows as the compact-open topology is finer than the p-topology.

Henceforth, Y^X will always have the compact-open topology. From Lemma 1, we see that if Y is a metric nondegenerate AE for normal spaces, and X is a compact space with $|X| > 2^{\aleph_0}$, Y^X is not even an ANE for perfectly normal spaces. For $d(Y^X) > \aleph_0$, but Y^X is metric, and E. Michael has shown that metric ANE's for Bing's perfectly normal, but not uncountably collectionwise normal, spaces must be separable [**6**, Theorem 3.1]. (Incidentally, this shows the "only if" parts of Problems 5.7 and 5.8 of Chapter XV of Dugundji's text [**2**] are incorrect.) We generalize this as follows:

LEMMA 2. *If Y is a metric ANE for Z, Z is $d(Y)$-collectionwise normal.*

PROOF (CF. PROPOSITION 5.1 OF [**6**]). Since Y is metric, its cellularity is attained and equals its density [**1**, Theorem 4]. Given any discrete collection $\{K_\alpha\}$ of no more than $d(Y)$ closed subsets of Z, we may then take as many disjoint nonempty open subsets $\{U_\alpha\}$ of Y. Let f be the continuous extension over an open neighborhood of $\bigcup K_\alpha$ of the map from

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$\bigcup K_\alpha$ to Y sending each K_α to a point in the corresponding U_α. Then $\{f^{-1}(U_\alpha)\}$ is a collection of disjoint open sets in Z separating $\{K_\alpha\}$.

THEOREM. *Let X be compact*, $\log |X| \geqq \aleph$. *Then if $Z \times X$ is normal, Z is $\aleph$-collectionwise normal. If Z is also $\aleph$-metacompact or semimetric, it is then $\aleph$-paracompact.*

PROOF. Let A be a closed subset of Z. If $f: A \to \boldsymbol{R}^X$ is continuous, so is its associate $f^*: A \times X \to \boldsymbol{R}$. We may extend f^* to a continuous map $F^*: Z \times X \to \boldsymbol{R}$, and its associate $F: Z \to \boldsymbol{R}^X$ is a continuous extension of f. Thus $\boldsymbol{R}^X$ is an AE for Z; by Lemma 1, $d(\boldsymbol{R}^X) \geqq \aleph$; and now by Lemma 2, Z is $\aleph$-collectionwise normal.

Bearing in mind that point-finite refinements may be made precise [**2**, Theorem VIII 1.4], one may adapt Michael's proof of Theorem 2 in [**5**] to show spaces both $\aleph$-metacompact and $\aleph$-collectionwise normal are $\aleph$-paracompact. Similarly, adapt McAuley's proof of Lemma 2 in [**4**] to handle the case where Z is semimetric.

Note if $\log |X| \geqq 2^{d(Z)}$ in the above, we get Z is collectionwise normal, for Z has no discrete collection of more than $2^{d(Z)}$ subsets [**2**, VII 3, Example 3].

Morita has shown Z is $\aleph$-paracompact and normal iff $Z \times I^\aleph$ is normal [**7**, Theorem 2.4]. The above theorem generalizes the "iff" part of this, replacing $I^\aleph$ with any compact space of equal cardinality, but draws a weaker conclusion.

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