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AN EXAMPLE IN FIXED POINT THEORY

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ABSTRACT. An example is given of a homeomorphism of R^2 onto R^2 which has no fixed points, such that each of its iterates has a fixed point.

A few years ago A. Granas posed the question: Does there exist a compact metric space X and a mapping (i.e. continuous function) $f: X \to X$ with no fixed points, such that each iterate f^n ($n \geqq 2$) of f has a fixed point?

W. Kuperberg has given as a simple solution (unpublished) the following: X is the set of all points on the Cartesian plane whose polar coordinates (p, θ) satisfy the inequality $1 \leqq p \leqq 2$ and $f: X \to X$ by

$$f(p, \theta) = (p, \theta + (2 + p)\pi/3).$$

It is easy to see that f is a homeomorphism of X onto X and that f has no fixed points. Oberve also that if n is an integer greater than 1, then any point on the circle $p = p_n$ is a fixed point for f^n where $p_n = 6K(n)/n - 2$ and $K(n)$ is an integer such that $\frac{1}{2} \leqq K(n)/n \leqq \frac{2}{3}$.

Our example gives an affirmative answer to the question: Is there a homeomorphism h of the Cartesian plane π onto itself which has no fixed point such that each iterate h^n ($n \geqq 2$) of h has a fixed point?

To this end, define the homeomorphism g from π onto π by $g(x, y) = (-x, y)$, let $W = \{(x, y) : |x| \leqq 8\}$, and define f on $\pi - W$ to be the identity function.

We now show how to define f on W such that $h = f \circ g$ is a homeomorphism of π onto π with the desired properties.

As an aid to this end, consider Figures 1 through 5.

In Figure 2 we indicate the image of a typical interval $[a, h]$ under f.

In Figure 3 we note that we require $c \to c'$, in Figure 4 that $b \to b'$, and in Figure 5 that c maps onto c'. These are the essential actions of f on W.

In Figure 1 we indicate the action of f on a portion of W and note that $h^3(P_3) = P_3$, $h^4(P_4) = P_4$ and $h^5(P_5) = P_5$.

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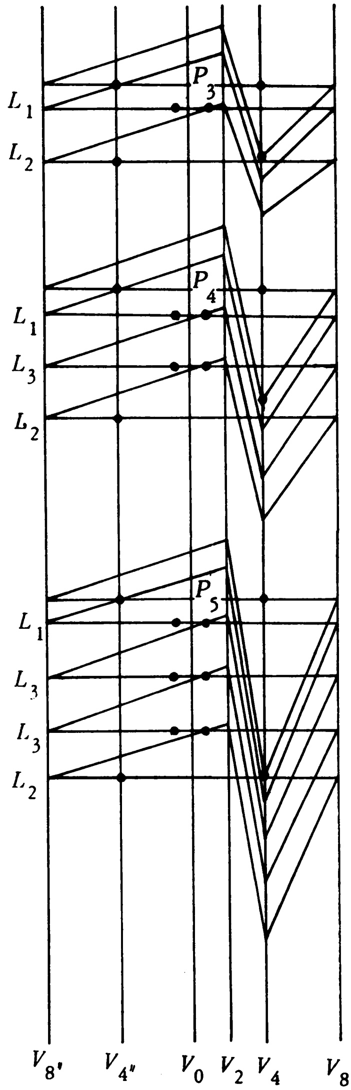

FIGURE 1

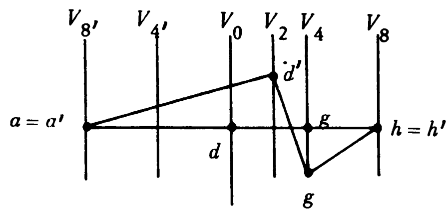

FIGURE 2

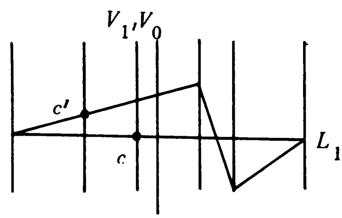

FIGURE 3

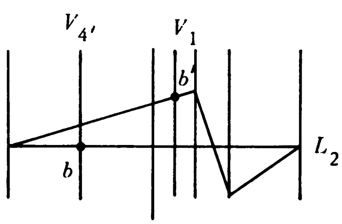

FIGURE 4

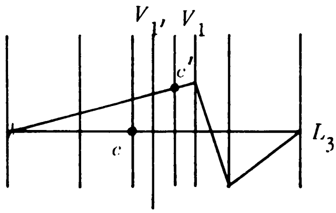

FIGURE 5

It is clear that f can be defined on W such that $h = f \circ g$ has the desired properties.

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