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AN ANNULAR FUNCTION WHICH IS THE SUM OF TWO NORMAL FUNCTIONS

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ABSTRACT. It is known that a nonconstant normal function cannot have a Koebe value. An example is presented of an annular function which is the sum of two normal holomorphic functions. This example shows that a sum of two normal functions can result in a nonconstant function which has the Koebe value ∞.

1. Let D denote the unit disc and C the unit circle in the complex plane. A function f meromorphic in D is said to be a *normal function* if the family of functions $\{f(S(z)):S\in\mathscr{S}\}$ is a normal family, where $\mathscr{S}$ is the collection of all conformal mappings of D onto itself. Let N denote the set of positive integers and let Z denote the set of all integers. A function f which is meromorphic in D is said to have the *Koebe value* α if there exists a sequence of Jordan arcs $\{J_n\}$ in D such that

(i) there exist distinct radii R_1 and R_2 of D such that, for each $n\in N$ the arc J_n meets both R_1 and R_2;

(ii) if $m_n=\inf\{|z|:z\in J_n\}$, then $m_n\to 1$ as $n\to\infty$; and

(iii) if α is finite and if $u_n=\sup\{|f(z)-\alpha|:z\in J_n\}$ then $u_n\to 0$ as $n\to\infty$; while if $\alpha=\infty$ and if $v_n=\inf\{|f(z)|:z\in J_n\}$, then $v_n\to\infty$ as $n\to\infty$.

A function f holomorphic in D is called an *annular* function if there exists a sequence of Jordan curves $\{J_n\}$ in D such that for each $n\in N$ the curve J_n is in the interior of J_{n+1} and conditions (ii) and (iii) are satisfied with $\alpha=\infty$.

It has been proved by Bagemihl and Seidel [**1**, Theorem 1, p. 10] that a nonconstant normal meromorphic function cannot have a Koebe value, either finite or infinite. The author [**2**, Theorem 5, p. 191] has shown that a sum of two normal functions need not result in a normal function. Further, since the Nevanlinna characteristic function $T(r)$ for a normal function satisfies $T(r)=O(\log(1/(1-r))$, the sum of two normal holomorphic functions must result in a function which is either a constant or is in Mac Lane's class A (see [**4**, pp. 43–44]), and such a function cannot have a finite Koebe value. We prove here that a sum of two normal homomorphic functions actually can have ∞ as a Koebe value in the strongest possible sense.

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2. If Γ is a crosscut of D with $0 \notin \Gamma$ and if w is a point of C we say that Γ *separates* 0 *from* w if the component of $D-\Gamma$ which contains 0 does not have w as a boundary point. The following lemma is basic.

LEMMA. *Let h be a function holomorphic in D. If for each point $w \in C$ and each $n \in N$ there exists a crosscut $\bigwedge(w, n)$ such that $|h(z)| \geqq n$ for each $z \in \bigwedge(w, n)$ and $\bigwedge(w, n)$ separates 0 from w, then h is an annular function.*

PROOF. For each $n \in N$ let D_n be the component of $\{z \in D : |h(z)| < n\}$ containing the origin. (It is possible that $D_n = \varnothing$ for a finite number of n's.) Let w be any point of C. For each $n \in N$, $\bigwedge(w, n) \cap D_n = \varnothing$ so that, if K_n denotes the boundary of D_n we have that $w \notin K_n$. Let n_0 be the least positive integer n for which $D_n \neq \varnothing$ and set $J_n = K_{n_0}$ for $n \leqq n_0$ and $J_n = K_n$ for $n > n_0$. Then J_n is a Jordan curve, $|h(z)| \geqq n$ for $z \in J_n$, and $C \cap J_n = \varnothing$. Thus h is an annular function.

THEOREM. *There exist two normal holomorphic functions f and g in D such that the sum $h = f + g$ is an annular function. Further, g can be chosen to be a function with positive real part.*

The proof of this theorem involves many detailed considerations, but the idea is basically to choose two normal holomorphic functions f and g such that the sum h satisfies the hypotheses of the Lemma. We note that, by renumbering the crosscuts $\bigwedge(w, n)$, if necessary, we can replace the condition that $|h(z)| \geqq n$ on $\bigwedge(w, n)$ by the more general condition that for each fixed $w \in C$ we have

$$\lim_{n \to \infty}(\inf\{|h(z)| : z \in \bigwedge(w, n)\}) = \infty. \tag{1}$$

Let $M(z)$ denote the elliptic modular function in D. Since $M(z)$ omits the values 0, 1, and ∞ in D, we can take $\log M(z)$ and $\log \log M(z)$ to be holomorphic in D by assigning an appropriate value to the origin. Further, since both $\log M(z)$ and $\log \log M(z)$ omit a countable set of values, both are normal functions. We shall be concerned with the real part of the logarithm function, and we use the notation $\mathrm{Re}(\log z) = \ln |z|$. Let A_1, A_2, A_3, and A_4 be the subsets of C defined by

$$\begin{aligned} A_1 &= \{w \in C : M(z) \text{ has an asymptotic value either 0 or } \infty \text{ at } w\}, \\ A_2 &= \{w \in C : \log M(z) \text{ has the asymptotic value 0 at } w\}, \\ A_3 &= \{w \in C : \text{for each } n \in N \text{ there exists a crosscut } \\ &\qquad \bigwedge(w, n) \text{ of } D \text{ which separates 0 from } w \text{ and such} \\ &\qquad\qquad \text{that } \ln|\log M(z)| > \ln n\pi \text{ for } z \in \bigwedge(w, n)\}, \end{aligned}$$

and

$$A_4 = C - (A_1 \cup A_2 \cup A_3).$$

We may assume, without loss of generality, that $1 \in A_2$. We note that both A_1 and A_2 are countable sets. Since the components of the set of z defined by $\operatorname{Arg} M(z)=k\pi$, $k \in N$, are all crosscuts of D which are arcs of circles orthogonal to C, and since the radial projection of these components onto C consists of all but a subset of C of measure zero, it follows that A_3 has measure 2π so that A_4 has measure zero (see [**3**, p. 123] for a more detailed account of this situation).

We claim that $A_1 \subset A_3$. If $w \in A_1$, then for each $n \in Z$ there exists a crosscut $\alpha(w, n)$ of D such that $\operatorname{Arg} M(z)=n\pi$ for $z \in \alpha(w, n)$ and $\alpha(w, n)$ is an arc of a circle orthogonal to C at w. For each $n \in N$, let G_n be the component of $D-(\alpha(w, n) \cup \alpha(w, -n))$ having both $\alpha(w, n)$ and $\alpha(w, -n)$ as boundary arcs of G_n. Then $|\ln|M(z)|| \to \infty$ as $z \to w$ from within G_n. Thus, by taking an appropriate crosscut of G_n, and combining this crosscut with appropriate subarcs of $\alpha(w, n)$ and $\alpha(w, -n)$ in the obvious manner, we obtain a crosscut $\Lambda(w, n)$ of D which separates 0 from w and is such that $\ln|\log M(z)| > \ln n\pi$ for $z \in \Lambda(w, n)$. Since the process can be done for each $n \in N$, we have that $w \in A_3$.

Now, for each $n \in N$, let E_n be the component of $\{z \in D: \ln|\log M(z)| < \ln n\pi\}$ which contains the origin. (It is possible that $E_n=\varnothing$ for a finite number of n.) If B_n is the collection of all points of C which are accessible boundary points of E_n, then B_n is a closed subset of C (see [**3**, Lemma, p. 121]). Clearly, for each $n \in N$, $B_n \cap A_3=\varnothing$. It is possible that $A_2 \cap B_n \neq \varnothing$ for some n, but we claim that the set $B_n'=B_n-A_2$ is a closed set for each $n \in N$. For if $w \in A_2$, then for each $n \in Z$ there exists a crosscut $\mathscr{B}(w, n)$ of D such that $\mathscr{B}(w, n)$ has one endpoint at w, $\mathscr{B}(w, n)$ is an arc of a circle orthogonal to C at w, and $\operatorname{Arg} \log M(z)=n\pi$ for $z \in \mathscr{B}(w, n)$. Thus, for each $n \in N$ and $z \in \mathscr{B}(w, n) \cup \mathscr{B}(w, -n)$ we have the $\ln|\log M(z)| > \ln n\pi$ so that if $w \in B_n \cap A_2$, then w is an isolated point of B_n. It follows that B_n' is a closed set for each $n \in N$. It should be further noted that for $w \in A_2$, $n \in Z$, the endpoint of $\mathscr{B}(w, n)$ other than w is a point of A_1, and hence is not a point of $B'_{|n|}$.

If $w \in A_2$, $n \in N$, and if H_n is the component of $D-(\mathscr{B}(w, n) \cup \mathscr{B}(w, -n))$ having both $\mathscr{B}(w, n)$ and $\mathscr{B}(w, -n)$ as boundary arcs, then

$$\ln|\log M(z)| \to -\infty$$

as $z \to w$ from within H_n. By taking an appropriate crosscut of H_n and combining this with appropriate subarcs of $\mathscr{B}(w, n)$ and $\mathscr{B}(w, -n)$ in the obvious manner, we obtain a crosscut $\Lambda(w, n)$ of D which separates 0 from w and which has both its endpoints in A_1 such that $|\log \log M(z)| > \ln n\pi$ for $z \in \Lambda(w, n)$.

Since A_2 is a countable set, let $A_2=\{w_1, w_2, w_3, \cdots, w_n, \cdots\}$ and let

$$P_n = \bigcup_{j=1}^{n} \left(\bigcup_{k=1}^{\infty} \Lambda(w_j, k) \right) \cup \{w_1, w_2, w_3, \cdots, w_n\},$$

and let

$$Q_n = P_n \cup \{z \in D: |z| \leqq 1 - 1/n\}.$$

Then Q_n is a compact subset in the closure of D, B'_n is a compact subset of C, and $Q_n \cap B'_n = \varnothing$. Let δ_n denote the distance between Q_n and B'_n, let $\delta_n(\theta)$ denote the distance from $e^{i\theta}$ to Q_n, and let $I_n = \{\theta \in (0, 2\pi): \delta_n(\theta) > \delta_n/2\}$. We note that

$$\int_{I_n} d\theta/\delta_n(\theta) < \infty.$$

We will make repeated use of the following general principle. If Q is a compact subset in the closure of D and if A is an arc of C where

$$A = \{e^{i\theta}: \theta_1 < \theta < \theta_2\}$$

such that $Q \cap A = \varnothing$, then for $z \in Q$ we have

$$\log \frac{e^{i\theta_1} - z}{e^{i\theta_2} - z} = \int_{\theta_2}^{\theta_1} \frac{ie^{i\theta}}{e^{i\theta} - z}\, d\theta \tag{2}$$

where the logarithm is taken with imaginary part between 0 and 2π for $z \in Q$. From (2), it follows that

$$\left| \log \frac{e^{i\theta_1} - z}{e^{i\theta_2} - z} \right| \leqq \int_{\theta_1}^{\theta_2} \frac{d\theta}{|e^{i\theta} - z|} = \int_{\theta_1}^{\theta_2} d\theta/\delta(\theta), \tag{3}$$

where $\delta(\theta)$ denotes the distance from $e^{i\theta}$ to Q.

Returning to the sets Q_n and B'_n, $n \in N$, and setting $S_n = \{\theta \in (0, 2\pi): e^{i\theta} \in B'_n\}$—recall that $w = 1 \in A_2$ and hence is not in B'_n—then $S_n \subset I_n$ and S_n is a compact set with measure zero since $B'_n \subset A_4$. Hence, for each $n \in N$ there exist a finite number of disjoint open intervals $(a(n,j), b(n,j))$, $1 \leqq j \leqq m_n$, such that

$$S_n \subset \bigcup_{j=1}^{m} (a(n,j), b(n,j)) \subset I_n,$$

$$\sum_{j=1}^{m_n} \int_{a(n,j)}^{b(n,j)} d\theta/\delta_n(\theta) < (1/2)^{n+1} \tag{4}$$

$$\sum_{j=1}^{m_n} (b(n,j) - a(n,j)) < (1/2)^{n+1}, \tag{5}$$

and such that none of the points $e^{ia(n,j)}$ or $e^{ib(n,j)}$ lies in A_2. By (3) and (4) we have that, for $n \in N$, $z \in Q_n$,

$$\sum_{j=1}^{m_n} \left| \log \frac{e^{ia(n,j)} - z}{e^{ib(n,j)} - z} \right| < (1/2)^{n+1}. \tag{6}$$

Now let

$$g_{n,j}(z) = (i/\pi) \log \frac{e^{ia(n,j)} - z}{e^{ib(n,j)} - z} + (1/2)(b(n,j) - a(n,j)).$$

Then $g_{n,j}(z)$ has the property that its real part is the harmonic measure of the arc $T_{n,j} = \{e^{i\theta} : a(n,j) < \theta < b(n,j)\}$. It follows that $\operatorname{Re} g_{n,j}(z) > 0$ for all $z \in D$, and $\operatorname{Re} g_{n,j}(z) \to 1$ as z approaches any point of $T_{n,j}$ from within D. We define

$$g(z) = \sum_{n=1}^{\infty} \sum_{j=1}^{m_n} g_{n,j}(z). \tag{7}$$

By (5), (6), and the fact that $D \subset \bigcup Q_n$, we have that $g(z)$ is a function holomorphic in D, and $g(z)$ has positive real part in D with $\operatorname{Re} g(z) \to +\infty$ as z approaches any point of A_4, since any point of A_4 is in all but a finite number of the sets B_n'. Further, for $z \in Q_n$, by (5), (6), and (7) we have that

$$|g(z)| \leqq 1 + \sup_{z \in Q_n} \sum_{k=1}^{n-1} \sum_{j=1}^{m_n} |g_{k,j}(z)|$$

where the right side is a finite sum and is thus a fixed number for each fixed n. It follows that if $w \in A_2$, then $w = w_q$ for some natural number q so that $w = w_q \in Q_n$ for $n \geqq q$ and this yields that $|\log \log M(z) + g(z)| \to \infty$ as $z \to w$ along $\bigcup_{k=1}^{\infty} \Lambda(w, k)$. Since $g(z)$ has positive real part, for $z \in A_3$ we have that $|\log \log M(z) + g(z)| \to \infty$ as $z \to w$ along $\bigcup_{k=1}^{\infty} \Lambda(w, k)$. It remains to obtain the same relationship for $w \in A_4$. However, we note that if $X = D - \{z \in D : |\log M(z)| < \frac{1}{2}\}$, then each point of A_4 is an accessible boundary point of X, and further, for each point $w \in A_4$ and each $n \in N$ there exists a crosscut $\Lambda(w, n)$ of D such that $\Lambda(w, n) \subset X \cap \{z \in D : |z - w| < 1/n\}$. (The region X is simply D with some nearly circular regions deleted, where each of the regions deleted has only one point of contact with C and each such point of contact is a point of A_2.) Each such $\Lambda(w, n)$ separates 0 from w. But $\ln|\log M(z)| > \ln(\frac{1}{2})$ for $z \in X$, and hence, since $\operatorname{Re} g(z) \to +\infty$ as $z \to w$ we have that $|\log \log M(z) + g(z)| \to \infty$ as $z \to w$ along $\bigcup_{k=1}^{\infty} \Lambda(w, k)$. Thus (1) is established for each $w \in C$ and, by the second sentence following the statement of the Theorem, together with the Lemma, the Theorem is proved with $f(z) = \log \log M(z)$ and $g(z)$ as defined in (7).

We remark that, since g is a function with positive real part it is a function of bounded type, so that the main burden of making h an annular function falls on f. For if $\{J_n\}$ is the sequence of Jordan curves for h satisfying (i), (ii), and (iii) for $\alpha=\infty$, and if Y_n denotes the region bounded by J_n for each n, then it is easy to show that if $\{\lambda_n\}$ is any sequence of real numbers tending to ∞, then the harmonic measure of the set $\{z \in J_n: |g(z)| \geqq \lambda_n\}$ relative to Y_n tends to zero as n tends to ∞. Thus, the harmonic measure of $\{z \in J_n: |f(z)| \geqq \lambda_n\}$ relative to Y_n tends to 2π as $n \to \infty$ for some choice of λ_n. Thus f is "almost" an annular function.

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