

PL INVOLUTIONS ON LENS SPACES AND OTHER 3-MANIFOLDS¹

PAIK KEE KIM

ABSTRACT. Let h be an involution of a 3-dimensional lens space $L=L(p, q)$. h is called sense preserving if h induces the identity of $H_1(L)$. The purpose of this paper is to classify the orientation preserving PL involutions of L which preserve sense and have nonempty fixed point sets for p even. It follows that, up to PL equivalences, there are exactly three PL involutions on the projective 3-space P^3 , and exactly seven PL involutions on $P^3\#P^3$.

1. Introduction. Throughout this paper, all spaces and maps are in the PL category. An involution h of a lens space $L=L(p, q)$ is called sense preserving if h induces the identity of $H_1(L)$. Kwun [3], [4] classified all orientation reversing involutions of L ($\neq S^3$) and all orientation preserving involutions of $L(p, q)$, p odd, which preserve sense and have nonempty fixed point sets. In this paper, we will investigate all orientation preserving involutions of $L(p, q)$, p even, which preserve sense and have nonempty fixed point sets.

Now consider free Z_2 -action h on P^3 . The orbit space M of h is a closed 3-manifold. Since we have a universal covering projection $S^3 \rightarrow P^3 \rightarrow M$, the order of $\Pi_1(M)$ is 4, and $\Pi_1(M) = Z_2 \oplus Z_2$ or Z_4 . Epstein [1] completely determined all possible abelian groups which can be fundamental groups of closed 3-manifolds; Z , $Z \oplus Z \oplus Z$, $Z \oplus Z_2$, and Z_r . Hence $\Pi_1(M)$ should be Z_4 . Hence $S^3/Z_4 = M$. Rice [9] discussed free Z_4 -action on S^3 . As a consequence of the discussion, $M=L(4, 1)$, and the classification problem for free Z_2 -actions on P^3 is essentially settled. It will be shown that, up to PL equivalences, there are exactly three PL involutions on P^3 .

Let M_i ($i=1, 2$) be 3-manifolds and h_i be involutions on M_i . If there is a suitable invariant 3-cell in each M_i , by taking the connected sum $M_1\#M_2$ along the 3-cells, one can define an involution, denoted by $h_1\#h_2$, of $M_1\#M_2$ induced by h_1 and h_2 . Notice that $h_1\#h_2$ depends on the choice of the invariant 3-cells along whose boundaries the connected

Received by the editors May 1, 1973 and, in revised form, September 11, 1973.

AMS (MOS) subject classifications (1970). Primary 55C35; Secondary 57A10.

Key words and phrases. PL involution, invariant set, fixed point set, lens space.

¹ Supported in part by NSF Grant GP 29515X. This paper is a part of the author's dissertation written under Professor K. W. Kwun and submitted to Michigan State University in 1973.

© American Mathematical Society 1974

sum is constructed. All orientation reversing involutions on $L(p, q) \# L(p', q')$ are known (see [2], [7], [10]). We will also investigate all orientation preserving PL involutions h on $P^3 \# P^3$. It follows that, up to PL equivalences, there are exactly seven PL involutions on $P^3 \# P^3$.

2. Fixed point sets. Let h be an orientation preserving PL involution on a lens space $L=L(p, q)$ which preserves sense and has nonempty fixed point set F . By the dimensional parity theorem, each component F_0 of F is of 1-dimension. Let U be a regular neighborhood of F_0 such that $U \cap F = F_0$. Consider the usual covering projection $g: S^3 \rightarrow L$. By the lifting theorem, we have a PL involution $\tilde{h}: (S^3, y_0) \rightarrow (S^3, y_0)$ where $g(y_0) \in F_0$. Since h is sense preserving, $g^{-1}(F_0)$ is connected, and $\tilde{F} = g^{-1}(F_0)$ is the fixed point set of \tilde{h} . By Waldhausen [13], \tilde{F} is an unknotted simple closed curve. Hence $\text{Cl}(S^3 - g^{-1}(U))$ is a solid torus, and $\text{Cl}(L - U)$ is a solid torus. An explicit argument of the above may be found in [4].

REMARK 2.1. Let D^2 be the unit disk in the Gaussian plane of complex numbers and S^1 its boundary. $D^2 \times S^1$ is a solid torus whose points can be denoted by $(\rho z_1, z_2)$ where $z_1, z_2 \in S^1$ and $0 \leq \rho \leq 1$. By using Stallings' result [11], one can show that the orbit space of a free PL involution h_0 on $D^2 \times S^1$ is homeomorphic to a disk bundle over S^1 , and h_0 is PL equivalent to an involution h_1 given by either $h_1(\rho z_1, z_2) = (\rho z_1, -z_2)$ or $h_1(\rho z_1, z_2) = (\rho \bar{z}_1, -z_2)$. It is known [12] that any orientation preserving PL involution on $D^2 \times S^1$ with nonempty fixed point set is PL equivalent to the involution h_2 on $D^2 \times S^1$ given by $h_2(\rho z_1, z_2) = (-\rho z_1, z_2)$, and the orbit space of h_2 is a solid torus.

THEOREM 2.2. *If h is an orientation preserving PL involution of $L=L(p, q)$, p even, which preserves sense and has nonempty fixed point set F , then F is a disjoint union of two simple closed curves.*

PROOF. By the above discussion, $L = D^2 \times S^1 \cup_k S^1 \times D^2$ such that $D^2 \times S^1$ is an invariant regular neighborhood of a component of F for an attaching map k of $S^1 \times S^1$. Suppose the contrary that $h|_{S^1 \times D^2}$ were free. Then by Remark 2.1, we may assume that $L = D^2 \times S^1 \cup_f S^1 \times D^2$ and h is given by $h(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$ and $h(z_1, \rho z_2) = (-z_1, \rho z_2)$ on $S^1 \times D^2$ for an appropriate attaching map f of $S^1 \times S^1$. Let $(1, 0)$ and $(0, 1)$ be the canonical generators of $\Pi_1(S^1 \times S^1)$ such that $f_{\#}(1, 0) = (a, b)$ and $f_{\#}(0, 1) = (c, d)$, where $f_{\#}$ is the automorphism induced by f (we disregard the base point as $\Pi_1(S^1 \times S^1, *)$ is abelian). We may assume that $|\begin{vmatrix} a & c \\ b & d \end{vmatrix}| = 1$. One can show that by Van Kampen's theorem,

$$\Pi_1(L) = \{\alpha, \beta \mid \beta^c = \alpha, \beta^a = 1\} = \{\beta \mid \beta^a = 1\}$$

where α and β are the canonical generators of $\Pi_1(D^2 \times S^1)$ and

$\Pi_1(S^1 \times D^2)$, respectively. Since $\Pi_1(L(p, q)) = \mathbb{Z}_p$, $a = \pm p$, and a is even. Let g and \hat{g} be the orbit maps of $h|_{D^2 \times S^1}$ and $h|_{S^1 \times D^2}$, respectively. Then by Remark 2.1, $g(D^2 \times S^1)$ and $\hat{g}(S^1 \times D^2)$ are solid tori. Consider the following diagram

$$\begin{array}{ccccc} D^2 \times S^1 \supset S^1 \times S^1 & \xrightarrow{f} & S^1 \times S^1 \subset S^1 \times D^2 & & \\ & \downarrow g' & & \downarrow \hat{g}' & \\ D^2 \times S^1 \supset S^1 \times S^1 & \xrightarrow{f'} & S^1 \times S^1 \subset S^1 \times D^2 & & \end{array}$$

where g' and \hat{g}' are induced by g and \hat{g} , respectively and f' is the induced attaching map in the orbit space of h . Notice that $g'_{\#}(r, s) = (2r, s)$ and $\hat{g}'_{\#}(r, s) = (2r, s)$ for any element $(r, s) \in \Pi_1(S^1 \times S^1)$. Let $f'_{\#}[(1, 0)] = (a', b')$ and $f'_{\#}[(0, 1)] = (c', d')$. By chasing the above commutative diagram, easy computation shows that $b = 2b'$, and b is even. Since a is even, we have a contradiction to the fact $ad - bc = 1$. Therefore, $\text{Fix}(h|_{S^1 \times D^2})$ cannot be empty. This completes the theorem.

REMARK 2.3. By Remark 2.1, we may assume that $L = D^2 \times S^1 \cup_f S^1 \times D^2$ and h is given by $h(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$ and $h(z_1, \rho z_2) = (z_1, -\rho z_2)$ on $S^1 \times D^2$ for an appropriate attaching map f of $S^1 \times S^1$.

3. **Involutions on $L(p, q)$.** Let $(1, 0)$ and $(0, 1)$ be the canonical generators of $\Pi_1(S^1 \times S^1)$ and k be a PL homeomorphism of $S^1 \times S^1$ such that $k_{\#}[(1, 0)] = (a, b)$ and $k_{\#}[(0, 1)] = (c, d)$. We may assume $|\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}| = 1$ and $a \geq 0$.

DEFINITION 3.1. Define $L_k(a, c, b, d) = D^2 \times S^1 \cup_k S^1 \times D^2$ where $|\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}| = 1$ and $a \geq 0$. We sometimes denote $L_k(a, c, b, d)$ by L_k if no confusion arises.

By Mangler [5], the isotopy classes of homeomorphisms of $S^1 \times S^1$ are precisely the automorphism classes of $\Pi_1(S^1 \times S^1)$. Hence the integers a, b, c and d completely determine the isotopy class of k in Definition 3.1, and hence the homeomorphic type of $L_k(a, c, b, d)$. As Kwun [4] pointed out, if $a = 0$, $L_k \approx S^1 \times S^2$, if $a = 1$, $L_k \approx S^3$, and if $a > 1$, $L_k \approx L(a, b)$. Recall that $L(p, q) \approx L(p, q')$ if and only if $q \equiv \pm q'$ or $qq' \equiv \pm 1 \pmod p$ [6], [8].

LEMMA 3.2. Let h be a PL involution of $L_k(a, c, b, d)$ such that $h(D^2 \times S^1) = D^2 \times S^1$ and h is given by $h(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$ and $h(z_1, \rho z_2) = (z_1, -\rho z_2)$ on $S^1 \times D^2$. Then the orbit space of h is homeomorphic to $L_{k'}(a/2, c, b, 2d)$, and a is even, where k' is the attaching map induced by k .

PROOF. By Remark 2.1, the orbit spaces of $h|_{D^2 \times S^1}$ and $h|_{S^1 \times D^2}$ are solid tori, and that of h is homeomorphic to $L_{k'}(a', c', b', d')$ for

suitable k', a', c', b' and d' . By computing in the same way as in Theorem 2.2, one can show that $a=2a', b=b', c=c',$ and $2d=d'$.

DEFINITION 3.3. Let p be even and a homeomorphism f of $S^1 \times S^1$ be given by $f(z_1, z_2) = (z_1^p z_2^c, z_1^b z_2^d)$. Define an involution of $L_f(p, c, b, d)$ by $h(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$ and $h(z_1, \rho z_2) = (z_1, -\rho z_2)$ on $S^1 \times D^2$. We denote the involution by $h(p, c, b, d)$.

In the above definition, since p is even, and b is odd, one can easily check that $h=h(p, c, b, d)$ is compatible with the attaching map f , i.e., $fh=hf$.

LEMMA 3.4. Let h' be any involution of $L_k(p, c, b, d)$ such that $h'(D^2 \times S^1) = D^2 \times S^1$, and $h'(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$ and $h'(z_1, \rho z_2) = (z_1, -\rho z_2)$ on $S^1 \times D^2$. Then h' is PL equivalent to $h=h(p, c, b, d)$.

PROOF. By Lemma 3.2, the orbit spaces of h and h' are $L_{f'}(p/2, c, b, 2d)$ and $L_{k'}(p/2, c, b, 2d)$, respectively. Therefore, f' and k' are isotopic, and there exists a PL homeomorphism $t': L_{k'} \rightarrow L_{f'}$ such that $t'(\rho z_1, z_2) = (\rho z_1, z_2)$ on $D^2 \times S^1$. Therefore, one can obtain a PL equivariant t by lifting t' .

REMARK 3.5. By Remark 2.3 and the above lemma, we may assume that every orientation preserving PL involution h of $L(p, q)$, p even, which preserves sense and has nonempty fixed point set is $h(p, c, b, d)$ for suitable c, b, d . Since $L_f(p, c, b, d) \approx L(p, b)$, $b \equiv \pm q$ or $bq \equiv \pm 1 \pmod p$. By Lemma 3.2, the orbit space of h is homeomorphic to $L(p/2, b)$ where $b \equiv \pm q$ or $bq \equiv \pm 1 \pmod p$.

PROPOSITION 3.6. $h=h(p, c, b, d)$ can be extended to an effective circle action.

PROOF. For each $z \in S^1$, define S^1 -action by $z \cdot (\rho z_1, z_2) = (\rho z_1 z, z_2)$ on $D^2 \times S^1$ and $z \cdot (z_1, \rho z_2) = (z_1 z^p, \rho z_2 z^b)$ on $S^1 \times D^2$.

REMARK 3.7. If an involution h of $L(p, q)$ can be extended to an effective circle action, h must be clearly sense preserving. By Proposition 3.6, $h(p, c, b, d)$ is sense preserving. Therefore, by Remark 3.5, the classification problem of orientation preserving PL involutions of $L(p, q)$, p even, which preserve sense and have nonempty fixed point sets is the same problem as the classification of those $h(p, c, b, d)$ for various possible c, b, d with $pd - cb = 1$.

Now we analyze the involutions $h(p, c, b, d)$. If $h(p, c, b, d)$ is equivalent to $h(p, c', b', d')$, we denote the fact by $h(p, c, b, d) \sim h(p, c', b', d')$.

LEMMA 3.8. For any integers c, b, d with $pd - cb = 1$,

- (1) $h(p, c, b, d) \sim h(p, c', b, d')$ for any integers c' and d' with $pd' - c'b = 1$.

- (2) $h(p, c, b, d) \sim h(p, c, b+mp, d+mc)$.
 (3) $h(p, c, b, d) \sim h(p, -c, -b, d)$.
 (4) $h(p, c, b, d) \sim h(p, -b, -c, d)$.

PROOF. We will define a homeomorphism $t: L_f \rightarrow L_{f'}$ where $L_f = L_f(p, c, b, d)$ and $L_{f'}$ is the space corresponding to the equivalent involution claimed in (i), $i=1, 2, 3, 4$. In (1), since $pd-bc=1=pd'-bc'$, $c'=c+mp$ and $d'=d+mb$ for some integer m . Define $t: L_f \rightarrow L_{f'}$ by $t(\rho z_1, z_2) = (\rho z_1 z_2^m, z_2)$ on $D^2 \times S^1$ and $t(z_1, \rho z_2) = (z_1, \rho z_2)$ on $S^1 \times D^2$. For (2), define $t: L_f \rightarrow L_{f'}$ by $t(\rho z_1, z_2) = (\rho z_1, z_2)$ on $D^2 \times S^1$ and $t(z_1, \rho z_2) = (z_1, \rho z_2 z_1^m)$ on $S^1 \times D^2$. For (3), define $t: L_f \rightarrow L_{f'}$ by $t(\rho z_1, z_2) = (\rho z_1, z_2^{-1})$ on $D^2 \times S^1$ and $t(z_1, \rho z_2) = (z_1, \rho z_2^{-1})$ on $S^1 \times D^2$. For (4), define $t: L_f \rightarrow L_{f'}$ by $t(\rho z_1, z_2) = (z_2, \rho z_1)$ on $D^2 \times S^1$ and $t(z_1, \rho z_2) = (\rho z_2, z_1)$ on $S^1 \times D^2$ such that $t(D^2 \times S^1) = S^1 \times D^2$. It is checked that those t are well defined and equivariant homeomorphisms. This completes the proof.

Now we are in a position to state our main theorem.

THEOREM 3.9. *Up to PL equivalences, there is exactly one orientation preserving PL involution on $L(p, q)$, p even, which preserves sense and has nonempty fixed point sets.*

PROOF. By Remark 3.7, we will consider two involutions $h_1 = h(p, c, b, d)$ and $h_2 = h(p, c', b', d')$. Let $L_1 = L_f(p, c, b, d)$ and $L_2 = L_{f'}(p, c', b', d')$ corresponding to h_1 and h_2 , respectively. Since $L_1 \approx L(p, b)$ and $L_2 \approx L(p, b')$, $b \equiv \pm b'$ or $bb' \equiv \pm 1 \pmod{p}$. If $b \equiv \pm b' \pmod{p}$, $b = \pm b' + mp$ for some integer m . By (1), (2), and (3), $h_1 \sim h_2$. Suppose $bb' \equiv \pm 1 \pmod{p}$. Since $pd-bc=1$, $b' \equiv \pm c \pmod{p}$, and $b' = \pm c + mp$ for some m . By (1), (2), (3), and (4), again $h_1 \sim h_2$. This completes the proof.

Obviously every involution of the projective 3-space P^3 is sense preserving. Kwun [3] showed that there is exactly one orientation reversing PL involution on P^3 , up to PL equivalences. Therefore, by Theorem 3.9 and some remark in the Introduction, we have:

THEOREM 3.10. *Up to PL equivalences, there are exactly three PL involutions on P^3 .*

4. PL involutions on $P^3 \# P^3$. Let M_i ($i=1, 2$) be oriented, connected, closed, irreducible 3-manifolds. It is known [2] that a PL involution h on $M_1 \# M_2$ is either the obvious involution which interchanges M_1 and M_2 or of the form $h_1 \# h_2$ where each h_i is a PL involution on M_i . In the latter case, if $\dim \text{Fix}(h) = 1$, the 2-sphere along which the M_1 and M_2 are joined meets F in general position. Obviously, if M_1 is not homeomorphic to M_2 , h is always of the form $h_1 \# h_2$. Notice that if h is of the

form $h_1\#h_2$, $\text{Fix}(h) \neq \emptyset$. Now suppose that $M_1=L(p, q)$ and $M_2=L(\bar{p}, \bar{q})$, and h is of the form $h_1\#h_2$. It will be convenient to call h decomposed sense preserving if h induces the identity of $H_1(M)=H_1(M_1)\oplus H_1(M_2)$. Obviously, if h is decomposed sense preserving, each h_i is sense preserving. Suppose that h_1 is sense preserving, $\dim \text{Fix}(h_1)=1$ and p is even. If $L(p, q)$ is symmetric (i.e., $q^2 \equiv \pm 1 \pmod p$), we claim that there exists a PL equivariant homeomorphism t on $L(p, q)$ (with respect to h_1) such that t interchanges the two components of $\text{Fix}(h_1)$. By Theorem 3.9, we may assume that $h_1=h_1(p, c, b, d)$ on $L_f(p, c, b, d)$. Since $L(p, q)$ is symmetric, $b^2 \equiv \pm 1 \pmod p$. Since $pd-cb=1$, $c = \pm b + mp$ for some integer m . By Lemma 3.8, we have the following equivariant maps t_i .

$$\begin{aligned} h_1(p, c, b, d) &= h_1(p, \pm b + mp, b, d) \stackrel{t_1}{\sim} h_1(p, b, \pm b + mp, d) \\ &\stackrel{t_2}{\sim} h_1(p, b, \pm b, d - mb) \stackrel{t_3}{\sim} h(p, \pm b, b, d - mb) \\ &\stackrel{t_4}{\sim} h(p, \pm b + mp, b, d) \\ &= h(p, c, b, d). \end{aligned}$$

Recall that $t_1(D^2 \times S^1) = S^1 \times D^2$ and $t_i(D^2 \times S^1) = D^2 \times S^1$ for $i \neq 1$. Let $t = t_4 t_3 t_2 t_1$. Then t is a PL equivariant homeomorphism on L_f such that $t(0 \times S^1) = S^1 \times 0$. This implies that $h_1\#h_2$ does not depend on how an invariant 3-cell of $L(p, q)$ is chosen to construct $h_1\#h_2$. Therefore, the following theorem is obtained by Theorem 3.9 and Kwun's result [4].

THEOREM 4.1. *Up to PL equivalences, there exists exactly one decomposed sense preserving PL involution h on $L(p, q)\#L(\bar{p}, \bar{q})$, which preserves the orientation if $L(p, q)$ and $L(\bar{p}, \bar{q})$ are symmetric (p, \bar{p} are any integers). There exist exactly two such h if $L(p, q)$ is symmetric and $L(\bar{p}, \bar{q})$ is a nonsymmetric lens space with \bar{p} odd (p is any integer).*

Since any involution on $P^3\#P^3$ of the form $h_1\#h_2$ is decomposed sense preserving, we have the following theorem.

THEOREM 4.2. *Let h be an orientation preserving PL involution on $P^3\#P^3$. If $\text{Fix}(h) = \emptyset$ or $\text{Fix}(h)$ is connected, h is the obvious involution which interchanges the two P^3 . If $\text{Fix}(h)$ is not connected, $\text{Fix}(h)$ is a disjoint union of three simple closed curves and there is exactly one such h , up to PL equivalences.*

It is known [2], [7], [10] that there exist exactly four orientation reversing PL involutions on $P^3\#P^3$ up to PL equivalences. Since $L(p, q)$, $p > 2$, does not admit an orientation reversing PL involution [3], any PL involution h on $L(p, q)\#L(\bar{p}, \bar{q})$ of the form $h_1\#h_2$ must be orientation preserving if p or $\bar{p} > 2$.

REFERENCES

1. D. B. A. Epstein, *Finite presentations of groups and 3-manifolds*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 205–212. MR **26** #1867.
2. P. K. Kim and J. F. Tollefson, *PL involutions on 3-manifolds* (to appear).
3. K. W. Kwun, *Scarcity of orientation-reversing PL involutions of lens spaces*, Michigan Math. J. **17** (1970), 355–358. MR **43** #5535.
4. ———, *Sense preserving PL involutions of some lens spaces*, Michigan Math. J. **20** (1973), 73–77.
5. W. Mangler, *Die Klassen von topologischen Abbildungen einer geschlossenen Fläche auf sich*, Math. Z. **44** (1939), 541–554.
6. E. Moise, *Affine structures in 3-manifolds. V. The triangulation theorem and Hauptvermutung*, Ann. of Math. (2) **56** (1952), 96–114. MR **14**, 72.
7. M. M. Myung, *PL involutions of some 3-manifolds*, Proc. Amer. Math. Soc. **33** (1972), 576–581. MR **45** #4429.
8. K. Reidemeister, *Homotopieringe und Linsenräume*, Abh. Math. Sem. Univ. Hamburg. **11** (1935), 102–109.
9. P. M. Rice, *Free actions of Z_4 on S^3* , Duke Math. J. **36** (1969), 749–751. MR **40** #2064.
10. D. Showers, Thesis, Michigan State University, East Lansing, Mich., 1973.
11. J. Stallings, *On fibering certain 3-manifolds*, Topology of 3-Manifolds and Related Topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 95–100. MR **28** #1600.
12. J. L. Tollefson, *Involutions on $S^1 \times S^2$ and other 3-manifolds*, Trans. Amer. Math. Soc. **183** (1973), 139–152.
13. F. Waldhausen, *Über Involutionen der 3-Sphäre*, Topology **8** (1969), 81–91. MR **38** #5209.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48823