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PL INVOLUTIONS ON LENS SPACES AND OTHER 3-MANIFOLDS[1]

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ABSTRACT. Let h be an involution of a 3-dimensional lens space $L=L(p,q)$. h is called sense preserving if h induces the identity of $H_1(L)$. The purpose of this paper is to classify the orientation preserving PL involutions of L which preserve sense and have nonempty fixed point sets for p even. It follows that, up to PL equivalences, there are exactly three PL involutions on the projective 3-space P^3, and exactly seven PL involutions on $P^3\#P^3$.

1. **Introduction.** Throughout this paper, all spaces and maps are in the PL category. An involution h of a lens space $L=L(p,q)$ is called sense preserving if h induces the identity of $H_1(L)$. Kwun [3], [4] classified all orientation reversing involutions of L ($\neq S^3$) and all orientation preserving involutions of $L(p,q)$, *p odd*, which preserve sense and have nonempty fixed point sets. In this paper, we will investigate all orientation preserving involutions of $L(p,q)$, *p even*, which preserve sense and have nonempty fixed point sets.

Now consider free Z_2-action h on P^3. The orbit space M of h is a closed 3-manifold. Since we have a universal covering projection $S^3\to P^3\to M$, the order of $\Pi_1(M)$ is 4, and $\Pi_1(M)=Z_2\oplus Z_2$ or Z_4. Epstein [1] completely determined all possible abelian groups which can be fundamental groups of closed 3-manifolds; Z, $Z\oplus Z\oplus Z$, $Z\oplus Z_2$, and Z_r. Hence $\Pi_1(M)$ should be Z_4. Hence $S^3/Z_4=M$. Rice [9] discussed free Z_4-action on S^3. As a consequence of the discussion, $M=L(4,1)$, and the classification problem for free Z_2-actions on P^3 is essentially settled. It will be shown that, up to PL equivalences, there are exactly three PL involutions on P^3.

Let M_i ($i=1,2$) be 3-manifolds and h_i be involutions on M_i. If there is a suitable invariant 3-cell in each M_i, by taking the connected sum $M_1\#M_2$ along the 3-cells, one can define an involution, denoted by $h_1\#h_2$, of $M_1\#M_2$ induced by h_1 and h_2. Notice that $h_1\#h_2$ depends on the choice of the invariant 3-cells along whose boundaries the connected

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sum is constructed. All orientation reversing involutions on $L(p,q)\#L(p',q')$ are known (see [2], [7], [10]). We will also investigate all orientation preserving PL involutions h on $P^3\#P^3$. It follows that, up to PL equivalences, there are exactly seven PL involutions on $P^3\#P^3$.

2. **Fixed point sets.** Let h be an orientation preserving PL involution on a lens space $L=L(p,q)$ which preserves sense and has nonempty fixed point set F. By the dimensional parity theorem, each component F_0 of F is of 1-dimension. Let U be a regular neighborhood of F_0 such that $U\cap F=F_0$. Consider the usual covering projection $g:S^3\to L$. By the lifting theorem, we have a PL involution $\tilde{h}:(S^3,y_0)\to(S^3,y_0)$ where $g(y_0)\in F_0$. Since h is sense preserving, $g^{-1}(F_0)$ is connected, and $\tilde{F}=g^{-1}(F_0)$ is the fixed point set of $\tilde{h}$. By Waldhausen [13], $\tilde{F}$ is an unknotted simple closed curve. Hence $\mathrm{Cl}(S^3-g^{-1}(U))$ is a solid torus, and $\mathrm{Cl}(L-U)$ is a solid torus. An explicit argument of the above may be found in [4].

REMARK 2.1. Let D^2 be the unit disk in the Gaussian plane of complex numbers and S^1 its boundary. $D^2\times S^1$ is a solid torus whose points can be denoted by $(\rho z_1, z_2)$ where $z_1, z_2\in S^1$ and $0\leqq\rho\leqq 1$. By using Stallings' result [11], one can show that the orbit space of a free PL involution h_0 on $D^2\times S^1$ is homeomorphic to a disk bundle over S^1, and h_0 is PL equivalent to an involution h_1 given by either $h_1(\rho z_1, z_2)=(\rho z_1, -z_2)$ or $h_1(\rho z_1, z_2)=(\rho\bar{z}_1, -z_2)$. It is known [12] that any orientation preserving PL involution on $D^2\times S^1$ with nonempty fixed point set is PL equivalent to the involution h_2 on $D^2\times S^1$ given by $h_2(\rho z_1, z_2)=(-\rho z_1, z_2)$, and the orbit space of h_2 is a solid torus.

THEOREM 2.2. *If h is an orientation preserving* PL *involution of $L=L(p,q)$, p even, which preserves sense and has nonempty fixed point set F, then F is a disjoint union of two simple closed curves.*

PROOF. By the above discussion, $L=D^2\times S^1\cup_k S^1\times D^2$ such that $D^2\times S^1$ is an invariant regular neighborhood of a component of F for an attaching map k of $S^1\times S^1$. Suppose the contrary that $h|S^1\times D^2$ were free. Then by Remark 2.1, we may assume that $L=D^2\times S^1\cup_f S^1\times D^2$ and h is given by $h(\rho z_1, z_2)=(-\rho z_1, z_2)$ on $D^2\times S^1$ and $h(z_1, \rho z_2)=(-z_1, \rho z_2)$ on $S^1\times D^2$ for an appropriate attaching map f of $S^1\times S^1$. Let $(1,0)$ and $(0,1)$ be the canonical generators of $\Pi_1(S^1\times S^1)$ such that $f_\#(1,0)=(a,b)$ and $f_\#(0,1)=(c,d)$, where $f_\#$ is the automorphism induced by f (we disregard the base point as $\Pi_1(S^1\times S^1, *)$ is abelian). We may assume that $\left|\begin{smallmatrix} a & c \\ b & d\end{smallmatrix}\right|=1$. One can show that by Van Kampen's theorem,

$$\Pi_1(L)=\{\alpha,\beta\mid \beta^c=\alpha,\ \beta^a=1\}=\{\beta\mid\beta^a=1\}$$

where α and β are the canonical generators of $\Pi_1(D^2\times S^1)$ and

$\Pi_1(S^1\times D^2)$, respectively. Since $\Pi_1(L(p,q))=z_p$, $a=\pm p$, and a is even. Let g and $\hat{g}$ be the orbit maps of $h|D^2\times S^1$ and $h|S^1\times D^2$, respectively. Then by Remark 2.1, $g(D^2\times S^1)$ and $\hat{g}(S^1\times D^2)$ are solid tori. Consider the following diagram

$$\begin{array}{ccccccc} D^2\times S^1 \supset & S^1\times S^1 & \xrightarrow{f} & S^1\times S^1 & \subset S^1\times D^2 \\ & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle \hat{g}'} & \\ D^2\times S^1 \supset & S^1\times S^1 & \xrightarrow{f'} & S^1\times S^1 & \subset S^1\times D^2 \end{array}$$

where g' and $\hat{g}'$ are induced by g and $\hat{g}$, respectively and f' is the induced attaching map in the orbit space of h. Notice that $g'_{\#}(r,s)=(2r,s)$ and $\hat{g}'_{\#}(r,s)=(2r,s)$ for any element $(r,s)\in\Pi_1(S^1\times S^1)$. Let $f'_{\#}[(1,0)]=(a',b')$ and $f'_{\#}[(0,1)]=(c',d')$. By chasing the above commutative diagram, easy computation shows that $b=2b'$, and b is even. Since a is even, we have a contradiction to the fact $ad-bc=1$. Therefore, $\mathrm{Fix}(h|S^1\times D^2)$ cannot be empty. This completes the theorem.

REMARK 2.3. By Remark 2.1, we may assume that $L=D^2\times S^1\cup_f S^1\times D^2$ and h is given by $h(\rho z_1,z_2)=(-\rho z_1,z_2)$ on $D^2\times S^1$ and $h(z_1,\rho z_2)=(z_1,-\rho z_2)$ on $S^1\times D^2$ for an appropriate attaching map f of $S^1\times S^1$.

3. Involutions on $L(p,q)$.

Let $(1,0)$ and $(0,1)$ be the canonical generators of $\Pi_1(S^1\times S^1)$ and k be a PL homeomorphism of $S^1\times S^1$ such that $k_{\#}[(1,0)]=(a,b)$ and $k_{\#}[(0,1)]=(c,d)$. We may assume $\left|\begin{smallmatrix} a & c\\ b & d\end{smallmatrix}\right|=1$ and $a\geq 0$.

DEFINITION 3.1. Define $L_k(a,c,b,d)=D^2\times S^1\cup_k S^1\times D^2$ where $\left|\begin{smallmatrix} a & c\\ b & d\end{smallmatrix}\right|=1$ and $a\geq 0$. We sometimes denote $L_k(a,c,b,d)$ by L_k if no confusion arises.

By Mangler **[5]**, the isotopy classes of homeomorphisms of $S^1\times S^1$ are precisely the automorphism classes of $\Pi_1(S^1\times S^1)$. Hence the integers a, b, c and d completely determine the isotopy class of k in Definition 3.1, and hence the homeomorphic type of $L_k(a,c,b,d)$. As Kwun **[4]** pointed out, if $a=0$, $L_k\approx S^1\times S^2$, if $a=1$, $L_k\approx S^3$, and if $a>1$, $L_k\approx L(a,b)$. Recall that $L(p,q)\approx L(p,q')$ if and only if $q\equiv\pm q'$ or $qq'\equiv\pm 1 \bmod p$ **[6]**, **[8]**.

LEMMA 3.2. *Let h be a* PL *involution of $L_k(a,c,b,d)$ such that $h(D^2\times S^1)=D^2\times S^1$ and h is given by $h(\rho z_1,z_2)=(-\rho z_1,z_2)$ on $D^2\times S^1$ and $h(z_1,\rho z_2)=(z_1,-\rho z_2)$ on $S^1\times D^2$. Then the orbit space of h is homeomorphic to $L_{k'}(a/2,c,b,2d)$, and a is even, where k' is the attaching map induced by k.*

PROOF. By Remark 2.1, the orbit spaces of $h|D^2\times S^1$ and $h|S^1\times D^2$ are solid tori, and that of h is homeomorphic to $L_{k'}(a',c',b',d')$ for suitable k', a', c', b' and d'. By computing in the same way as in Theorem 2.2, one can show that $a=2a'$, $b=b'$, $c=c'$, and $2d=d'$.

DEFINITION 3.3. Let p be even and a homeomorphism f of $S^1\times S^1$ be given by $f(z_1, z_2)=(z_1^p z_2^c, z_1^b z_2^d)$. Define an involution of $L_f(p, c, b, d)$ by $h(\rho z_1, z_2)=(-\rho z_1, z_2)$ on $D^2\times S^1$ and $h(z_1, \rho z_2)=(z_1, -\rho z_2)$ on $S^1\times D^2$. We denote the involution by $h(p, c, b, d)$.

In the above definition, since p is even, and b is odd, one can easily check that $h=h(p, c, b, d)$ is compatible with the attaching map f, i.e., $fh=hf$.

LEMMA 3.4. *Let h' be any involution of $L_k(p, c, b, d)$ such that $h'(D^2\times S^1)=D^2\times S^1$, and $h'(\rho z_1, z_2)=(-\rho z_1, z_2)$ on $D^2\times S^1$ and $h'(z_1, \rho z_2)=(z_1, -\rho z_2)$ on $S^1\times D^2$. Then h' is* PL *equivalent to $h=h(p, c, b, d)$.*

PROOF. By Lemma 3.2, the orbit spaces of h and h' are $L_{f'}(p/2, c, b, 2d)$ and $L_{k'}(p/2, c, b, 2d)$, respectively. Therefore, f' and k' are isotopic, and there exists a PL homeomorphism $t': L_{k'}\to L_{f'}$ such that $t'(\rho z_1, z_2)=(\rho z_1, z_2)$ on $D^2\times S^1$. Therefore, one can obtain a PL equivariant t by lifting t'.

REMARK 3.5. By Remark 2.3 and the above lemma, we may assume that every orientation preserving PL involution h of $L(p, q)$, p even, which preserves sense and has nonempty fixed point set is $h(p, c, b, d)$ for suitable c, b, d. Since $L_f(p, c, b, d)\approx L(p, b)$, $b\equiv\pm q$ or $bq\equiv\pm 1 \bmod p$. By Lemma 3.2, the orbit space of h is homeomorphic to $L(p/2, b)$ where $b\equiv\pm q$ or $bq\equiv\pm 1 \bmod p$.

PROPOSITION 3.6. *$h=h(p, c, b, d)$ can be extended to an effective circle action.*

PROOF. For each $z\in S^1$, define S^1-action by $z\cdot(\rho z_1, z_2)=(\rho z_1 z, z_2)$ on $D^2\times S^1$ and $z\cdot(z_1, \rho z_2)=(z_1 z^p, \rho z_2 z^b)$ on $S^1\times D^2$.

REMARK 3.7. If an involution h of $L(p, q)$ can be extended to an effective circle action, h must be clearly sense preserving. By Proposition 3.6, $h(p, c, b, d)$ is sense preserving. Therefore, by Remark 3.5, the classification problem of orientation preserving PL involutions of $L(p, q)$, p even, which preserve sense and have nonempty fixed point sets is the same problem as the classification of those $h(p, c, b, d)$ for various possible c, b, d with $pd-cb=1$.

Now we analyze the involutions $h(p, c, b, d)$. If $h(p, c, b, d)$ is equivalent to $h(p, c', b', d')$, we denote the fact by $h(p, c, b, d)\sim h(p, c', b', d')$.

LEMMA 3.8. *For any integers c, b, d with $pd-cb=1$,*

(1) *$h(p, c, b, d)\sim h(p, c', b, d')$ for any integers c' and d' with $pd'-c'b=1$.*

(2) $h(p, c, b, d) \sim h(p, c, b+mp, d+mc)$.
(3) $h(p, c, b, d) \sim h(p, -c, -b, d)$.
(4) $h(p, c, b, d) \sim h(p, -b, -c, d)$.

PROOF. We will define a homeomorphism $t: L_f \to L_{f'}$ where $L_f = L_f(p, c, b, d)$ and $L_{f'}$ is the space corresponding to the equivalent involution claimed in (i), $i=1, 2, 3, 4$. In (1), since $pd-bc=1=pd'-bc'$, $c'=c+mp$ and $d'=d+mb$ for some integer m. Define $t: L_f \to L_{f'}$ by $t(\rho z_1, z_2)=(\rho z_1 \bar{z}_2^{m}, z_2)$ on $D^2 \times S^1$ and $t(z_1, \rho z_2)=(z_1, \rho z_2)$ on $S^1 \times D^2$. For (2), define $t: L_f \to L_{f'}$ by $t(\rho z_1, z_2)=(\rho z_1, z_2)$ on $D^2 \times S^1$ and $t(z_1, \rho z_2)=(z_1, \rho z_2 z_1^m)$ on $S^1 \times D^2$. For (3), define $t: L_f \to L_{f'}$ by $t(\rho z_1, z_2)=(\rho z_1, z_2^{-1})$ on $D^2 \times S^1$ and $t(z_1, \rho z_2)=(z_1, \rho z_2^{-1})$ on $S^1 \times D^2$. For (4), define $t: L_f \to L_{f'}$ by $t(\rho z_1, z_2)=(z_2, \rho z_1)$ on $D^2 \times S^1$ and $t(z_1, \rho z_2)=(\rho z_2, z_1)$ on $S^1 \times D^2$ such that $t(D^2 \times S^1)=S^1 \times D^2$. It is checked that those t are well defined and equivariant homeomorphisms. This completes the proof.

Now we are in a position to state our main theorem.

THEOREM 3.9. *Up to* PL *equivalences, there is exactly one orientation preserving* PL *involution on $L(p, q)$, p even, which preserves sense and has nonempty fixed point sets.*

PROOF. By Remark 3.7, we will consider two involutions $h_1 = h(p, c, b, d)$ and $h_2=h(p, c', b', d')$. Let $L_1=L_f(p, c, b, d)$ and $L_2 = L_{f'}(p, c', b', d')$ corresponding to h_1 and h_2, respectively. Since $L_1 \approx L(p, b)$ and $L_2 \approx L(p, b')$, $b \equiv \pm b'$ or $bb' \equiv \pm 1 \bmod p$. If $b \equiv \pm b' \bmod p$, $b = \pm b' + mp$ for some integer m. By (1), (2), and (3), $h_1 \sim h_2$. Suppose $bb' \equiv \pm 1 \bmod p$. Since $pd-bc=1$, $b' \equiv \pm c \bmod p$, and $b' = \pm c+mp$ for some m. By (1), (2), (3), and (4), again $h_1 \sim h_2$. This completes the proof.

Obviously every involution of the projective 3-space P^3 is sense preserving. Kwun [3] showed that there is exactly one orientation reversing PL involution on P^3, up to PL equivalences. Therefore, by Theorem 3.9 and some remark in the Introduction, we have:

THEOREM 3.10. *Up to* PL *equivalences, there are exactly three* PL *involutions on P^3.*

4. PL **involutions on $P^3 \# P^3$.** Let M_i $(i=1, 2)$ be oriented, connected, closed, irreducible 3-manifolds. It is known [2] that a PL involution h on $M_1 \# M_2$ is either the obvious involution which interchanges M_1 and M_2 or of the form $h_1 \# h_2$ where each h_i is a PL involution on M_i. In the latter case, if dim Fix$(h)=1$, the 2-sphere along which the M_1 and M_2 are joined meets F in general position. Obviously, if M_1 is not homeomorphic to M_2, h is always of the form $h_1 \# h_2$. Notice that if h is of the

form $h_1\#h_2$, $\mathrm{Fix}(h)\neq\varnothing$. Now suppose that $M_1=L(p,q)$ and $M_2=L(\bar{p},\bar{q})$, and h is of the form $h_1\#h_2$. It will be convenient to call h decomposed sense preserving if h induces the identity of $H_1(M)=H_1(M_1)\oplus H_1(M_2)$. Obviously, if h is decomposed sense preserving, each h_i is sense preserving. Suppose that h_1 is sense preserving, $\dim \mathrm{Fix}(h_1)=1$ and p is even. If $L(p,q)$ is symmetric (i.e., $q^2\equiv\pm 1 \bmod p$), we claim that there exists a PL equivariant homeomorphism t on $L(p,q)$ (with respect to h_1) such that t interchanges the two components of $\mathrm{Fix}(h_1)$. By Theorem 3.9, we may assume that $h_1=h_1(p,c,b,d)$ on $L_f(p,c,b,d)$. Since $L(p,q)$ is symmetric, $b^2\equiv\pm 1 \bmod p$. Since $pd-cb=1$, $c=\pm b+mp$ for some integer m. By Lemma 3.8, we have the following equivariant maps t_i.

$$\begin{aligned} h_1(p,c,b,d) &= h_1(p,\pm b+mp,b,d) \overset{t_1}{\sim} h_1(p,b,\pm b+mp,d) \\ &\overset{t_2}{\sim} h_1(p,b,\pm b,d-mb) \overset{t_3}{\sim} h(p,\pm b,b,d-mb) \\ &\overset{t_4}{\sim} h(p,\pm b+mp,b,d) \\ &= h(p,c,b,d). \end{aligned}$$

Recall that $t_1(D^2\times S^1)=S^1\times D^2$ and $t_i(D^2\times S^1)=D^2\times S^1$ for $i\neq 1$. Let $t=t_4t_3t_2t_1$. Then t is a PL equivariant homeomorphism on L_f such that $t(0\times S^1)=S^1\times 0$. This implies that $h_1\#h_2$ does not depend on how an invariant 3-cell of $L(p,q)$ is chosen to construct $h_1\#h_2$. Therefore, the following theorem is obtained by Theorem 3.9 and Kwun's result [**4**].

THEOREM 4.1. *Up to* PL *equivalences, there exists exactly one decomposed sense preserving* PL *involution h on $L(p,q)\#L(\bar{p},\bar{q})$, which preserves the orientation if $L(p,q)$ and $L(\bar{p},\bar{q})$ are symmetric (p, $\bar{p}$ are any integers). There exist exactly two such h if $L(p,q)$ is symmetric and $L(\bar{p},\bar{q})$ is a nonsymmetric lens space with $\bar{p}$ odd (p is any integer).*

Since any involution on $P^3\#P^3$ of the form $h_1\#h_2$ is decomposed sense preserving, we have the following theorem.

THEOREM 4.2. *Let h be an orientation preserving* PL *involution on $P^3\#P^3$. If* $\mathrm{Fix}(h)=\varnothing$ *or* $\mathrm{Fix}(h)$ *is connected, h is the obvious involution which interchanges the two P^3. If* $\mathrm{Fix}(h)$ *is not connected,* $\mathrm{Fix}(h)$ *is a disjoint union of three simple closed curves and there is exactly one such h, up to* PL *equivalences.*

It is known [**2**], [**7**], [**10**] that there exist exactly four orientation reversing PL involutions on $P^3\#P^3$ up to PL equivalences. Since $L(p,q)$, $p>2$, does not admit an orientation reversing PL involution [**3**], any PL involution h on $L(p,q)\#L(\bar{p},\bar{q})$ of the form $h_1\#h_2$ must be orientation preserving if p or $\bar{p}>2$.

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