PL INVOLUTIONS ON LENS SPACES
AND OTHER 3-MANIFOLDS¹

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Abstract. Let \( h \) be an involution of a 3-dimensional lens space \( L=L(p,q) \). \( h \) is called sense preserving if \( h \) induces the identity of \( H_1(L) \). The purpose of this paper is to classify the orientation preserving PL involutions of \( L \) which preserve sense and have nonempty fixed point sets for \( p \) even. It follows that, up to PL equivalences, there are exactly three PL involutions on the projective 3-space \( P^3 \), and exactly seven PL involutions on \( P^3\#P^3 \).

1. Introduction. Throughout this paper, all spaces and maps are in the PL category. An involution \( h \) of a lens space \( L=L(p,q) \) is called sense preserving if \( h \) induces the identity of \( H_1(L) \). Kwun [3], [4] classified all orientation reversing involutions of \( L (\neq S^3) \) and all orientation preserving involutions of \( L(p,q) \), \( p \) odd, which preserve sense and have nonempty fixed point sets. In this paper, we will investigate all orientation preserving involutions of \( L(p,q) \), \( p \) even, which preserve sense and have nonempty fixed point sets.

Now consider free \( \mathbb{Z}_2 \)-action \( h \) on \( P^3 \). The orbit space \( M \) of \( h \) is a closed 3-manifold. Since we have a universal covering projection \( S^3\to P^3\to M \), the order of \( \Pi_1(M) \) is 4, and \( \Pi_1(M) = \mathbb{Z}_2\oplus\mathbb{Z}_2 \) or \( \mathbb{Z}_4 \). Epstein [1] completely determined all possible abelian groups which can be fundamental groups of closed 3-manifolds; \( \mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}, \mathbb{Z}\oplus\mathbb{Z}_2 \), and \( \mathbb{Z}_r \). Hence \( \Pi_1(M) \) should be \( \mathbb{Z}_4 \). Hence \( S^3/\mathbb{Z}_4=M \). Rice [9] discussed free \( \mathbb{Z}_4 \)-action on \( S^3 \). As a consequence of the discussion, \( M=L(4,1) \), and the classification problem for free \( \mathbb{Z}_2 \)-actions on \( P^3 \) is essentially settled. It will be shown that, up to PL equivalences, there are exactly three PL involutions on \( P^3 \).

Let \( M_i \) (\( i=1, 2 \)) be 3-manifolds and \( h_i \) be involutions on \( M_i \). If there is a suitable invariant 3-cell in each \( M_i \), by taking the connected sum \( M_1\#M_2 \) along the 3-cells, one can define an involution, denoted by \( h_1\#h_2 \), of \( M_1\#M_2 \) induced by \( h_1 \) and \( h_2 \). Notice that \( h_1\#h_2 \) depends on the choice of the invariant 3-cells along whose boundaries the connected

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sum is constructed. All orientation reversing involutions on $L(p, q)$ and $L(p', q')$ are known (see [2], [7], [10]). We will also investigate all orientation preserving PL involutions $h$ on $P^3 # P^3$. It follows that, up to PL equivalences, there are exactly seven PL involutions on $P^3 # P^3$.

2. Fixed point sets. Let $h$ be an orientation preserving PL involution on a lens space $L = L(p, q)$ which preserves sense and has nonempty fixed point set $F$. By the dimensional parity theorem, each component $F_0$ of $F$ is of 1-dimension. Let $U$ be a regular neighborhood of $F_0$ such that $U \cap F = F_0$. Consider the usual covering projection $g: S^3 \rightarrow L$. By the lifting theorem, we have a PL involution $\tilde{h}: (S^3, y_0) \rightarrow (S^3, y_0)$ where $g(y_0) \in F_0$. Since $h$ is sense preserving, $g^{-1}(F_0)$ is connected, and $F = g^{-1}(F_0)$ is the fixed point set of $\tilde{h}$. By Waldhausen [13], $F$ is an unknotted simple closed curve. Hence $\text{Cl}(S^3 - g^{-1}(U))$ is a solid torus, and $\text{Cl}(L - U)$ is a solid torus. An explicit argument of the above may be found in [4].

Remark 2.1. Let $D^2$ be the unit disk in the Gaussian plane of complex numbers and $S^1$ its boundary. $D^2 \times S^1$ is a solid torus whose points can be denoted by $(\rho z_1, z_2)$ where $z_1, z_2 \in S^1$ and $0 \leq \rho \leq 1$. By using Stallings' result [11], one can show that the orbit space of a free PL involution $h_0$ on $D^2 \times S^1$ is homeomorphic to a disk bundle over $S^1$, and $h_0$ is PL equivalent to an involution $h_1$ given by either $h_1(\rho z_1, z_2) = (\rho z_1, -z_2)$ or $h_1(\rho z_1, z_2) = (\rho z_1, -z_2)$. It is known [12] that any orientation preserving PL involution on $D^2 \times S^1$ with nonempty fixed point set is PL equivalent to the involution $h_2$ on $D^2 \times S^1$ given by $h_2(\rho z_1, z_2) = (-\rho z_1, z_2)$, and the orbit space of $h_2$ is a solid torus.

Theorem 2.2. If $h$ is an orientation preserving PL involution of $L = L(p, q)$, $p$ even, which preserves sense and has nonempty fixed point set $F$, then $F$ is a disjoint union of two simple closed curves.

Proof. By the above discussion, $L = D^2 \times S^1 \cup S^1 \times D^2$ such that $D^2 \times S^1$ is an invariant regular neighborhood of a component of $F$ for an attaching map $k$ of $S^1 \times S^1$. Suppose the contrary that $h|S^1 \times D^2$ were free. Then by Remark 2.1, we may assume that $L = D^2 \times S^1 \cup S^1 \times D^2$ and $h$ is given by $h(\rho z_1, z_2) = (-\rho z_1, z_2)$ on $D^2 \times S^1$ and $h(z_1, \rho z_2) = (-z_1, \rho z_2)$ on $S^1 \times D^2$ for an appropriate attaching map $f$ of $S^1 \times S^1$. Let $(1, 0)$ and $(0, 1)$ be the canonical generators of $\Pi_1(S^1 \times S^1)$ such that $f_#(1, 0) = (a, b)$ and $f_#(0, 1) = (c, d)$, where $f_#$ is the automorphism induced by $f$ (we disregard the base point as $\Pi_1(S^1 \times S^1, *)$ is abelian). We may assume that $|a| = 1$. One can show that by Van Kampen’s theorem,

$$\Pi_1(L) = \{\alpha, \beta \mid \beta^a = \alpha, \beta^a = 1\} = \{\beta \mid \beta^a = 1\}$$

where $\alpha$ and $\beta$ are the canonical generators of $\Pi_1(D^2 \times S^1)$ and
\(\Pi_1(S^1 \times D^2)\), respectively. Since \(\Pi_1(L(p,q)) = z_p\), \(a = \pm p\), and \(a\) is even.

Let \(g\) and \(\bar{g}\) be the orbit maps of \(h|D^2 \times S^1\) and \(h|S^1 \times D^2\), respectively. Then by Remark 2.1, \(g(D^2 \times S^1)\) and \(\bar{g}(S^1 \times D^2)\) are solid tori. Consider the following diagram

\[
\begin{array}{ccc}
D^2 \times S^1 & \rightarrow & S^1 \times S^1 \\
\downarrow g' & & \downarrow \bar{g}' \\
D^2 \times S^1 & \rightarrow & S^1 \times S^1
\end{array}
\]

where \(g'\) and \(\bar{g}'\) are induced by \(g\) and \(\bar{g}\), respectively, and \(f'\) is the induced attaching map in the orbit space of \(h\). Notice that \(g'_p(r,s) = (2r,s)\) and \(g'_p(r,s) = (2r,s)\) for any element \((r,s) \in \Pi_1(S^1 \times S^1)\). Let \(f'_p[(1,0)] = (a', b')\) and \(f'_p[(0,1)] = (c', d')\). By chasing the above commutative diagram, easy computation shows that \(b = 2b'\), and \(b\) is even. Since \(a\) is even, we have a contradiction to the fact \(ad - bc = 1\). Therefore, \(\text{Fix}(h|S^1 \times D^2)\) cannot be empty. This completes the theorem.

**Remark 2.3.** By Remark 2.1, we may assume that \(L = D^2 \times S^1 \cup_r S^1 \times D^2\) and \(h\) is given by \(h(\rho z_1, z_2) = (-\rho z_1, z_2)\) on \(D^2 \times S^1\) and \(h(z_1, \rho z_2) = (z_1, -\rho z_2)\) on \(S^1 \times D^2\) for an appropriate attaching map \(f\) of \(S^1 \times S^1\).

### 3. Involutions on \(L(p,q)\)

Let \((1,0)\) and \((0,1)\) be the canonical generators of \(\Pi_1(S^1 \times S^1)\) and \(k\) be a PL homeomorphism of \(S^1 \times S^1\) such that \(k_p[(1,0)] = (a, b)\) and \(k_p[(0,1)] = (c, d)\). We may assume \(|d| = 1\) and \(a \geq 0\).

**Definition 3.1.** Define \(L_k(a,c,b,d) = D^2 \times S^1 \cup_k S^1 \times D^2\) where \(\rho = 1\) and \(a \geq 0\). We sometimes denote \(L_k(a,c,b,d)\) by \(L_k\) if no confusion arises.

By Mangler [5], the isotopy classes of homeomorphisms of \(S^1 \times S^1\) are precisely the automorphism classes of \(\Pi_1(S^1 \times S^1)\). Hence the integers \(a, b, c\) and \(d\) completely determine the isotopy class of \(k\) in Definition 3.1, and hence the homeomorphic type of \(L_k(a,c,b,d)\). As Kwun [4] pointed out, if \(a = 0\), \(L_k \approx S^2\), if \(a = 1\), \(L_k \approx S^3\), and if \(a > 1\), \(L_k \approx L(a,b)\).

Recall that \(L(p,q) \approx L(p,q')\) if and only if \(q \equiv \pm q'\) or \(qq' \equiv \pm 1\) mod \(p\) [6], [8].

**Lemma 3.2.** Let \(h\) be a PL involution of \(L_k(a,c,b,d)\) such that \(h(D^2 \times S^1) = D^2 \times S^1\) and \(h\) is given by \(h(\rho z_1, z_2) = (-\rho z_1, z_2)\) on \(D^2 \times S^1\) and \(h(z_1, \rho z_2) = (z_1, -\rho z_2)\) on \(S^1 \times D^2\). Then the orbit space of \(h\) is homeomorphic to \(L_k'(a,2,c,b,2d)\), and \(a\) is even, where \(k'\) is the attaching map induced by \(k\).

**Proof.** By Remark 2.1, the orbit spaces of \(h|D^2 \times S^1\) and \(h|S^1 \times D^2\) are solid tori, and that of \(h\) is homeomorphic to \(L_k'(a', c', b', d')\) for
suitable $k', a', c', b'$ and $d'$. By computing in the same way as in Theorem 2.2, one can show that $a=2a', b=b'$, $c=c'$, and $2d=d'$.

**Definition 3.3.** Let $p$ be even and a homeomorphism $f$ of $S^1 \times S^1$ be given by $f(z_1, z_2) = (z_1^p z_2^c, z_1^b z_2^d)$. Define an involution of $L_f(p, c, b, d)$ by $h(pz_1, z_2) = (-pz_1, z_2)$ on $D^2 \times S^1$ and $h(z_1, pz_2) = (z_1, -pz_2)$ on $S^1 \times D^2$. We denote the involution by $h(p, c, b, d)$.

In the above definition, since $p$ is even, and $b$ is odd, one can easily check that $h = h(p, c, b, d)$ is compatible with the attaching map $f$, i.e., $fh = hf$.

**Lemma 3.4.** Let $h'$ be any involution of $L_h(p, c, b, d)$ such that $h'(D^2 \times S^1) = D^2 \times S^1$, and $h'(pz_1, z_2) = (-pz_1, z_2)$ on $D^2 \times S^1$ and $h'(z_1, pz_2) = (z_1, -pz_2)$ on $S^1 \times D^2$. Then $h'$ is PL equivalent to $h = h(p, c, b, d)$.

**Proof.** By Lemma 3.2, the orbit spaces of $h$ and $h'$ are $L_f(p/2, c, b, 2d)$ and $L_h(p/2, c, b, 2d)$, respectively. Therefore, $f'$ and $k'$ are isotopic, and there exists a PL homeomorphism $t': L_h \to L_f$ such that $t'(pz_1, z_2) = (pz_1, z_2)$ on $D^2 \times S^1$. Therefore, one can obtain a PL equivariant $t$ by lifting $t'$.

**Remark 3.5.** By Remark 2.3 and the above lemma, we may assume that every orientation preserving PL involution $h$ of $L(p, q)$, $p$ even, which preserves sense and has nonempty fixed point set is $h(p, c, b, d)$ for suitable $c, b, d$. Since $L_f(p, c, b, d) \approx L(p, b)$, $b \equiv \pm q$ or $bq \equiv \pm 1$ mod $p$. By Lemma 3.2, the orbit space of $h$ is homeomorphic to $L(p/2, b)$ where $b \equiv \pm q$ or $bq \equiv \pm 1$ mod $p$.

**Proposition 3.6.** $h = h(p, c, b, d)$ can be extended to an effective circle action.

**Proof.** For each $z \in S^1$, define $S^1$-action by $z \cdot (pz_1, z_2) = (pz_1z, z_2)$ on $D^2 \times S^1$ and $z \cdot (z_1, pz_2) = (z_1z^p, pz_2z^b)$ on $S^1 \times D^2$.

**Remark 3.7.** If an involution $h$ of $L(p, q)$ can be extended to an effective circle action, $h$ must be clearly sense preserving. By Proposition 3.6, $h(p, c, b, d)$ is sense preserving. Therefore, by Remark 3.5, the classification problem of orientation preserving PL involutions of $L(p, q)$, $p$ even, which preserve sense and have nonempty fixed point sets is the same problem as the classification of those $h(p, c, b, d)$ for various possible $c, b, d$ with $pd - cb = 1$.

Now we analyze the involutions $h(p, c, b, d)$. If $h(p, c, b, d)$ is equivalent to $h(p, c', b', d')$, we denote the fact by $h(p, c, b, d) \sim h(p, c', b', d')$.

**Lemma 3.8.** For any integers $c, b, d$ with $pd - cb = 1$,

1. $h(p, c, b, d) \sim h(p, c', b', d')$ for any integers $c'$ and $d'$ with $pd' - c'b = 1$. 

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(2) \( h(p, c, b, d) \sim h(p, c, b + mp, d + mc) \).
(3) \( h(p, c, b, d) \sim h(p, -c, -b, d) \).
(4) \( h(p, c, b, d) \sim h(p, -b, -c, d) \).

Proof. We will define a homeomorphism \( t : L_f \rightarrow L_f' \) where \( L_f = L_f(p, c, b, d) \) and \( L_f' \) is the space corresponding to the equivalent involution claimed in (i), \( i = 1, 2, 3, 4 \). In (1), since \( pd - bc = 1 = pd' - bc' \), \( c' = c + mp \) and \( d' = d + mb \) for some integer \( m \). Define \( t : L_f \rightarrow L_f \) by \( t(pz_1, z_2) = (pz_1z_2^{-m}, z_2) \) on \( D^2 \times S^1 \) and \( t(z_1, pz_2) = (z_1, pz_2) \) on \( S^1 \times D^2 \).

For (2), define \( t : L_f \rightarrow L_f' \) by \( t(pz_1, z_2) = (pz_1, z_2z_1) \) on \( D^2 \times S^1 \) and \( t(z_1, pz_2) = (z_1, pz_2z_1^m) \) on \( S^1 \times D^2 \). For (3), define \( t : L_f \rightarrow L_f' \) by \( t(pz_1, z_2) = (pz_1z_2^{-1}, z_2) \) on \( D^2 \times S^1 \) and \( t(z_1, pz_2) = (z_1, pz_2z_2^{-1}) \) on \( S^1 \times D^2 \).

For (4), define \( t : L_f \rightarrow L_f' \) by \( t(pz_1, z_2) = (z_2, pz_1) \) on \( D^2 \times S^1 \) and \( t(z_1, pz_2) = (pz_2, z_1) \) on \( S^1 \times D^2 \) such that \( t(D^2 \times S^1) = S^1 \times D^2 \). It is checked that those \( t \) are well defined and equivariant homeomorphisms. This completes the proof.

Now we are in a position to state our main theorem.

**Theorem 3.9.** Up to PL equivalences, there is exactly one orientation preserving PL involution on \( L(p, q) \), \( p \) even, which preserves sense and has nonempty fixed point sets.

Proof. By Remark 3.7, we will consider two involutions \( h_1 = h(p, c, b, d) \) and \( h_2 = h(p, c', b', d') \). Let \( L_1 = L_f(p, c, b, d) \) and \( L_2 = L_f(p, c', b', d') \) corresponding to \( h_1 \) and \( h_2 \), respectively. Since \( L_1 \cong L(p, b) \) and \( L_2 \cong L(p, b'), \) \( b \equiv \pm b' \mod p \) or \( b' \equiv \pm 1 \mod p \). If \( b \equiv \pm b' \mod p, b = \pm b' + mp \) for some integer \( m \). By (1), (2), and (3), \( h_1 \sim h_2 \).

Suppose \( bb' \equiv \pm 1 \mod p \). Since \( pd - bc = 1, b' \equiv \pm c \mod p, \) and \( b' = \pm c + mp \) for some \( m \). By (1), (2), (3), and (4), again \( h_1 \sim h_2 \). This completes the proof.

Obviously every involution of the projective 3-space \( P^3 \) is sense preserving. Kwun [3] showed that there is exactly one orientation reversing PL involution on \( P^3 \), up to PL equivalences. Therefore, by Theorem 3.9 and some remark in the Introduction, we have:

**Theorem 3.10.** Up to PL equivalences, there are exactly three PL involutions on \( P^3 \).

4. PL involutions on \( P^3 \# P^3 \). Let \( M_i \) (\( i = 1, 2 \)) be oriented, connected, closed, irreducible 3-manifolds. It is known [2] that a PL involution \( h \) on \( M_1 \# M_2 \) is either the obvious involution which interchanges \( M_1 \) and \( M_2 \) or of the form \( h_1 \# h_2 \) where each \( h_i \) is a PL involution on \( M_i \). In the latter case, if \( \dim \text{Fix}(h) = 1 \), the 2-sphere along which the \( M_1 \) and \( M_2 \) are joined meets \( F \) in general position. Obviously, if \( M_1 \) is not homeomorphic to \( M_2 \), \( h \) is always of the form \( h_1 \# h_2 \). Notice that if \( h \) is of the
form $h_1 \# h_2$, $\text{Fix}(h) \neq \emptyset$. Now suppose that $M_1 = L(p, q)$ and $M_2 = L(\bar{p}, \bar{q})$, and $h$ is of the form $h_1 \# h_2$. It will be convenient to call $h$ decomposed sense preserving if $h$ induces the identity of $H_1(M) = H_1(M_1) \oplus H_1(M_2)$. Obviously, if $h$ is decomposed sense preserving, each $h_i$ is sense preserving. Suppose that $h_1$ is sense preserving, $\dim \text{Fix}(h_1) = 1$ and $p$ is even. If $L(p, q)$ is symmetric (i.e., $q^2 \equiv \pm 1 \mod p$), we claim that there exists a PL equivariant homeomorphism $t$ on $L(p, q)$ (with respect to $h_1$) such that $t$ interchanges the two components of $\text{Fix}(h_1)$. By Theorem 3.9, we may assume that $h_1 = h_1(p, c, b, d)$ on $L(p, c, b, d)$. Since $L(p, q)$ is symmetric, $b^2 \equiv \pm 1 \mod p$. Since $pd - cb = 1$, $c = \pm b + mp$ for some integer $m$. By Lemma 3.8, we have the following equivariant maps $t_i$.

\[
\begin{align*}
    h_1(p, c, b, d) &\rightarrow h_1(p, b, \pm b + mp, d) \\
    &\sim h_1(p, b, \pm b, d - mb) \\
    &\sim h(p, \pm b + mp, b, d) \\
    &\Rightarrow h(p, c, b, d).
\end{align*}
\]

Recall that $t_1(D^2 \times S^1) = S^1 \times D^2$ and $t_i(D^2 \times S^1) = D^2 \times S^1$ for $i \neq 1$. Let $t = t_1 t_2 t_3 t_4$. Then $t$ is a PL equivariant homeomorphism on $L$, such that $t(0 \times S^1) = S^1 \times 0$. This implies that $h_1 \# h_2$ does not depend on how an invariant 3-cell of $L(p, q)$ is chosen to construct $h_1 \# h_2$. Therefore, the following theorem is obtained by Theorem 3.9 and Kwun's result [4].

**Theorem 4.1.** Up to PL equivalences, there exists exactly one decomposed sense preserving PL involution $h$ on $L(p, q) \# L(\bar{p}, \bar{q})$, which preserves the orientation if $L(p, q)$ and $L(\bar{p}, \bar{q})$ are symmetric ($p, \bar{p}$ are any integers). There exist exactly two such $h$ if $L(p, q)$ is symmetric and $L(\bar{p}, \bar{q})$ is a nonsymmetric lens space with $\bar{p}$ odd ($p$ is any integer).

Since any involution on $P^3 \# P^3$ of the form $h_1 \# h_2$ is decomposed sense preserving, we have the following theorem.

**Theorem 4.2.** Let $h$ be an orientation preserving PL involution on $P^3 \# P^3$. If $\text{Fix}(h) = \emptyset$ or $\text{Fix}(h)$ is connected, $h$ is the obvious involution which interchanges the two $P^3$. If $\text{Fix}(h)$ is not connected, $\text{Fix}(h)$ is a disjoint union of three simple closed curves and there is exactly one such $h$, up to PL equivalences.

It is known [2], [7], [10] that there exist exactly four orientation reversing PL involutions on $P^3 \# P^3$ up to PL equivalences. Since $L(p, q)$, $p > 2$, does not admit an orientation reversing PL involution [3], any PL involution $h$ on $L(p, q) \# L(\bar{p}, \bar{q})$ of the form $h_1 \# h_2$ must be orientation preserving if $p$ or $\bar{p} > 2$. 
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