ON THE UPPER BOUND OF THE NUMBER OF REAL
ROOTS OF A RANDOM ALGEBRAIC EQUATION
WITH INFINITE VARIANCE. II

G. SAMAL AND M. N. MISHRA

Abstract. Let $N_n$ be the number of real roots of \( \sum_{v=0}^{n} a_v \xi_v x^v = 0 \)
where $\xi_v$'s are independent random variables identically distributed
with a common characteristic function $\exp(-C|t|^\alpha)$; $C$ is a positive
constant, $a_0, a_1, \cdots, a_n$ are nonzero real numbers such that
\( k_n = \max_{0 \leq v \leq n} |a_v| = O(n^\beta / \log n) \). Then
(i) $\Pr\left( \sup_{n > n_0} N_n / (\log n)^2 > \mu \right) < \mu / n_0^{2\alpha - 2 - \beta}$, $1 \leq \alpha \leq 2$, $0 < \beta < 1$;
(ii) $\Pr\left( N_n / (\log n)^2 > \mu \right) < \mu / n$, $\alpha \geq 1$;
(iii) $\Pr\left( N_n / (\log n)^2 > \mu \right) < \mu / n^{3\alpha - 1 - \beta}$, $1 \leq \alpha \leq 2$.

1. Introduction. In our paper [2] we have considered the upper
bound of the number of real roots of the algebraic equation

\[ f(x) = \sum_{v=0}^{n} a_v \xi_v x^v = 0 \]

whose coefficients $\xi_v$'s are identically distributed independent random
variables with a common characteristic function $\exp(-C|t|^\alpha)$ where $C$
is a positive constant and $\alpha \geq 1$. In fact we have proved that

(i) $\Pr\left( \sup_{n > n_0} N_n / (\log n)^2 > \mu \right) < \mu / n_0^{2\alpha - 2 - \beta}$, $0 < \beta < 1$, $1 \leq \alpha \leq 2$;
(ii) $\Pr\left( N_n / (\log n)^2 > \mu \right) < \mu / n$, $\alpha \geq 1$;
(iii) $\Pr\left( N_n / (\log n)^2 > \mu \right) < \mu / n^{3\alpha - 1 - \beta}$, $1 \leq \alpha \leq 2$,

where $N_n$ is the number of real roots of $f(x) = 0$.

In the present work we like to point out that there is no need of taking
the coefficients as identically distributed and that our result could be
put in more general form.

In the place of equation (1) we shall consider the equation

\[ f(x) = \sum_{v=0}^{n} a_v \xi_v x^v = 0 \]
where \( a_0, a_1, \ldots, a_n \) are nonzero real numbers with some restrictions. The coefficients \( a_v \xi_v \)'s are not identically distributed although \( \xi_v \)'s have the same distribution.

Equations of the type (2) have been considered by Dunnage [1], but his variance is finite whereas our variance is infinite for \( 1 < \alpha < 2 \).

In our paper [3] we have considered the corresponding result for the lower bound.

2. THEOREM 1. Let \( f(x) = \sum_{v=0}^{n} a_v \xi_v x^v \) be a polynomial where \( \xi_v \)'s are independent random variables identically distributed with a common characteristic function \( \exp(-C|t|^\alpha) \), \( C \) is a positive constant and \( 1 \leq \alpha \leq 2 \). \( a_0, a_1, a_2, \ldots, a_n \) are nonzero real numbers such that \( k_n = \max_{0 \leq v \leq n} |a_v|, \ t_n = \min_{0 \leq v \leq n} |a_v| \). Then there exists an integer \( n_0 \) such that for each \( n > n_0 \), the number of real roots of the equation \( f(x) = 0 \) is at most \( \mu \log n)^2 \) except for a set of measure at most \( \mu \log n^{3-2-\beta} \).

PROOF. We shall indicate only the modifications necessary in the proof of Theorem 1 of our paper cited above. The steps not mentioned here remain unchanged.

In §5, we shall have

\[
\Pr\{|a_v \xi_v| \geq (n + 1)^3\} \leq \mu |a_v|^\alpha/(n + 1)^{3\alpha}.
\]

Hence

\[
\Pr\{|a_v \xi_v| < (n + 1)^3, 0 \leq v \leq n\} > 1 - (\mu \sum_{v=0}^{n} |a_v|^\alpha)/(n + 1)^{3\alpha}
\]

\[
> 1 - \mu k_n^\alpha/(n + 1)^{3\alpha - 1}.
\]

So outside a set of measure at most \( \mu k_n^\alpha/(n + 1)^{3\alpha - 1} \),

\[
\max_{|z| \leq 1 + \sqrt{2/n}} |f(z)| \leq C^\alpha(n + 1)^4.
\]

Now, \( \Pr\{\xi_v < 1/n^5\} \leq \mu/C^{1/\alpha} n^5 \). Since the characteristic function of \( f(x_m) \) is \( \exp(-C|t|^\alpha \sum_{v=0}^{n} |a_v|^\alpha x_m^v) \) we have

\[
\Pr\{|f(x_m)| < 1/n^5\} \leq \frac{\mu}{n^5} \left( \sum_{v=0}^{n} |a_v|^\alpha x_m^v \right)^{-1/\alpha} < \frac{\mu}{t_n n^5} \left( \sum_{v=0}^{n} x_m^v \right)^{-1/\alpha}
\]

for \( m = 1, 2, \ldots, k, p \log n \) and

\[
\Pr\{|f(x_0)| < 1/n^5\} \leq \frac{\mu}{n^5} \left( \sum_{v=0}^{n} |a_v|^\alpha \right)^{-1/\alpha} < \mu/(n^{5+1/\alpha} \cdot t_n)
\]

for \( m = 0 \).

So the number of zeros of \( f(z) \) in all the circles \( C_0, C_1, \ldots, C_k, C_p \log n \) is at most \( \mu \log n^2 \).
The measure of the exceptional set is at most
\[
\frac{\mu k_n^\alpha \log n}{(n + 1)^{3^\alpha - 1}} + \frac{\mu \log n}{t_n \cdot n^{5 + 1/\alpha}} + \frac{\mu}{n^{5 + 1/\alpha} \cdot t_n} < \frac{\mu k_n^\alpha \log n}{(n + 1)^{3^\alpha - 1}} + \frac{\mu \log n}{t_n \cdot n^{5 + 1/\alpha}} + \frac{\mu}{n^{5 + 1/\alpha} \cdot t_n}
\]
since \(k_n^\alpha = O(n^\beta / \log n)\) and without loss of generality we could suppose \(t_n \geq 1\). The measure of the exceptional set corresponding to the segment \([0, 1/3]\) is at most
\[
\mu k_n^\alpha (n + 1)^{3^\alpha - 1} + \mu /n^{3\alpha - 1 - \beta} = \mu /n^{3\alpha - 1 - \beta}.
\]

There is no change in the remaining part of the proof.

3. **THEOREM 2.** Let \(N_n\) be the number of real roots of the equation \(f(x) = 0\) satisfying the conditions of Theorem 1, where \(\alpha \geq 1\). Then \(N_n < \mu (\log n)^2\) outside a set of measure at most \(\mu /n\).

**PROOF.** In §6 of our paper,
\[
Pr\{\|a_v\| \geq (n + 1)^3\} \leq \mu |a_v|^\alpha (n + 1)^{3^\alpha}
\]
and
\[
Pr\{\|\xi_v\| < 1/n\} \leq \mu/C^{1/\alpha}.n.
\]

So the measure of the exceptional set is at most
\[
\frac{\mu k_n^\alpha \log n}{(n + 1)^{3^\alpha - 1}} + \frac{\mu \log n}{t_n \cdot n^{1 + 1/\alpha}} + \frac{\mu}{n^{1 + 1/\alpha} \cdot t_n} < \frac{\mu k_n^\alpha \log n}{(n + 1)^{3^\alpha - 1}} + \mu /t_n \cdot n < \mu /n^{3\alpha - 1 - \beta} + \mu /n < \mu /n
\]
since \(t_n \geq 1\) and \(k_n^\alpha = O(n^\beta / \log n)\).

**REFERENCES**


DEPARTMENT OF MATHEMATICS, RAVENSHAW COLLEGE, CUTTACK 3, ORISSA, INDIA

(Current address of G. Samal)

*Current address* (M. N. Mishra): Bureau of Statistics, Bhubaneswar, Orissa, India