PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 44, Number 2, June 1974

ON THE UPPER BOUND OF THE NUMBER OF REAL ROOTS OF A RANDOM ALGEBRAIC EQUATION WITH INFINITE VARIANCE. II

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ABSTRACT. Let N_n be the number of real roots of $\sum_{\nu=0}^{n} a_\nu \xi_\nu x^\nu = 0$ where ξ_ν's are independent random variables identically distributed with a common characteristic function $\exp(-C|t|^\alpha)$; C is a positive constant, $a_0, a_1, \cdots, a_n$ are nonzero real numbers such that $k_n = \max_{0\leqq\nu\leqq n} |a_\nu| = O(n^\beta/\log n)$. Then

(i) $\Pr\{\operatorname{Sup}_{n>n_0} N_n/(\log n)^2 > \mu\} < \mu'/n_0^{3\alpha-2-\beta}$, $1\leqq\alpha\leqq 2$, $0<\beta<1$;

(ii) $\Pr\{N_n/(\log n)^2 > \mu\} < \mu'/n$, $\alpha\geqq 1$;

(iii) $\Pr\{N_n/(\log n)^2 > \mu\} < \mu'/n^{3\alpha-1-\beta}$, $1\leqq\alpha\leqq 2$.

1. **Introduction.** In our paper [2] we have considered the upper bound of the number of real roots of the algebraic equation

$$f(x) = \sum_{\nu=0}^{n} \xi_\nu x^\nu = 0 \tag{1}$$

whose coefficients ξ_ν's are identically distributed independent random variables with a common characteristic function $\exp(-C|t|^\alpha)$ where C is a positive constant and $\alpha\geqq 1$. In fact we have proved that

(i) $\Pr\left\{\operatorname*{Sup}_{n>n_0} N_n/(\log n)^2 > \mu\right\} < \mu'/n_0^{3\alpha-2-\beta}$, $\quad 0<\beta<1, 1\leqq\alpha\leqq 2$;

(ii) $\qquad \Pr\{N_n/(\log n)^2 > \mu\} < \mu'/n$, $\qquad \alpha\geqq 1$;

(iii) $\qquad \Pr\{N_n/(\log n)^2 > \mu\} < \mu'/n^{3\alpha-1-\beta}$, $\qquad 1\leqq\alpha\leqq 2$,

where N_n is the number of real roots of $f(x)=0$.

In the present work we like to point out that there is no need of taking the coefficients as identically distributed and that our result could be put in more general form.

In the place of equation (1) we shall consider the equation

$$f(x) = \sum_{\nu=0}^{n} a_\nu \xi_\nu x^\nu = 0 \tag{2}$$

Received by the editors December 25, 1972 and, in revised form, September 6, 1973.
AMS (MOS) subject classifications (1970). Primary 60–XX.
Key words and phrases. Random variables, infinite variance, real roots, random algebraic equations.

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where $a_0, a_1, \cdots, a_n$ are nonzero real numbers with some restrictions. The coefficients $a_\nu \xi_\nu$'s are not identically distributed although ξ_ν's have the same distribution.

Equations of the type (2) have been considered by Dunnage [**1**], but his variance is finite whereas our variance is infinite for $1<\alpha<2$.

In our paper [**3**] we have considered the corresponding result for the lower bound.

2. THEOREM 1. *Let* $f(x)=\sum_{\nu=0}^{n} a_\nu \xi_\nu x^\nu$ *be a polynomial where* ξ_ν*'s are independent random variables identically distributed with a common characteristic function* $\exp(-C|t|^\alpha)$, *C is a positive constant and* $1\leqq\alpha\leqq 2$. $a_0, a_1, a_2, \cdots, a_n$ *are nonzero real numbers such that* $k_n^\alpha=O(n^\beta/\log n)$, $0<\beta<1$ *where* $k_n=\max_{0\leqq\nu\leqq n}|a_\nu|$, $t_n=\min_{0\leqq\nu\leqq n}|a_\nu|$. *Then there exists an integer* n_0 *such that for each* $n>n_0$, *the number of real roots of the equation* $f(x)=0$ *is at most* $\mu(\log n)^2$ *except for a set of measure at most* $\mu'/n_0^{3\alpha-2-\beta}$.

PROOF. We shall indicate only the modifications necessary in the proof of Theorem 1 of our paper cited above. The steps not mentioned here remain unchanged.

In §5, we shall have

$$\Pr\{|a_\nu \xi_\nu| \geqq (n+1)^3\} \leqq \mu\, |a_\nu|^\alpha/(n+1)^{3\alpha}.$$

Hence

$$\begin{aligned}\Pr\{|a_\nu\xi_\nu| < (n+1)^3, 0 \leqq \nu \leqq n\} &> 1-(\mu \textstyle\sum_0^n |a_\nu|^\alpha)/(n+1)^{3\alpha} \\ &> 1-\mu K_n^\alpha/(n+1)^{3\alpha-1}.\end{aligned}$$

So outside a set of measure at most $\mu K_n^\alpha/(n+1)^{3\alpha-1}$,

$$\max_{|z|\leqq 1+2/n} |f(z)| \leqq C^2(n+1)^4.$$

Now, $\Pr\{|\xi_\nu|<1/n^5\}\leqq\mu/C^{1/\alpha}n^5$. Since the characteristic function of $f(x_m)$ is $\exp(-C|t|^\alpha \sum_{\nu=0}^n |a_\nu|^\alpha x_m^{\alpha\nu})$ we have

$$\Pr\{|f(x_m)| < 1/n^5\} \leqq \frac{\mu}{n^5}\left(\sum_{\nu=0}^n |a_\nu|^\alpha x_m^{\alpha\nu}\right)^{-1/\alpha} < \frac{\mu}{t_n n^5}\left(\sum x_m^{\alpha\nu}\right)^{-1/\alpha}$$

for $m=1, 2, \cdots, k, p \log n$ and

$$\Pr\{|f(x_0)| < 1/n^5\} \leqq \frac{\mu}{n^5}\left(\sum_{\nu=0}^n |a_\nu|^\alpha\right)^{-1/\alpha} < \mu/(n^{5+1/\alpha}\cdot t_n)$$

for $m=0$.

So the number of zeros of $f(z)$ in all the circles $C_0, C_1, \cdots, C_k, C_{p\log n}$ is at most $\mu(\log n)^2$.

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The measure of the exceptional set is at most

$$\frac{\mu k_n^\alpha \log n}{(n+1)^{3\alpha-1}}+\frac{\mu}{t_n\cdot n^5}\sum_{m=1}^{\nu\log n}\left(\sum_{\nu=0}^{n}x_m^{\alpha\nu}\right)^{-1/\alpha}+\frac{\mu}{n^{5+1/\alpha}\cdot t_n}$$

$$<\frac{\mu k_n^\alpha \log n}{(n+1)^{3\alpha-1}}+\frac{\mu\log n}{t_n\cdot n^{5+1/\alpha}}+\frac{\mu}{t_n\cdot n^5}+\frac{\mu}{n^{5+1/\alpha}\cdot t_n}$$

$$<\frac{\mu k_n^\alpha \log n}{(n+1)^{3\alpha-1}}+\frac{\mu}{t_n\cdot n^5}<\frac{\mu}{n^{3\alpha-1-\beta}},$$

since $k_n^\alpha=O(n^\beta/\log n)$ and without loss of generality we could suppose $t_n\geqq 1$. The measure of the exceptional set corresponding to the segment $[0,\frac{1}{2}]$ is at most

$$\mu k_n^\alpha/(n+1)^{3\alpha-1}+\mu/n^5<\mu/n^{3\alpha-1-\beta}.$$

There is no change in the remaining part of the proof.

3. THEOREM 2. *Let N_n be the number of real roots of the equation $f(x)=0$ satisfying the conditions of Theorem* 1, *where $\alpha\geqq 1$. Then $N_n<\mu(\log n)^2$ outside a set of measure at most μ'/n.*

PROOF. In §6 of our paper,

$$\Pr\{|a_\nu\xi_\nu|\geqq(n+1)^3\}\leqq\mu\,|a_\nu|^\alpha/(n+1)^{3\alpha}$$

and

$$\Pr\{|\xi_\nu|<1/n\}\leqq\mu/C^{1/\alpha}n.$$

So the measure of the exceptional set is at most

$$\frac{\mu k_n^\alpha \log n}{(n+1)^{3\alpha-1}}+\frac{\mu}{t_n}\frac{\log n}{n^{1+1/\alpha}}+\frac{\mu}{t_n\cdot n}+\frac{\mu}{n^{1+1/\alpha}\cdot t_n}$$

$$<\mu k_n^\alpha\log n/(n+1)^{3\alpha-1}+\mu/(t_n\cdot n)$$

$$<\mu/n^{3\alpha-1-\beta}+\mu/n<\mu/n$$

since $t_n\geqq 1$ and $k_n^\alpha=O(n^\beta/\log n)$.

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