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CORRECTIONS TO "AN ANALYTICAL CRITERION FOR THE COMPLETENESS OF RIEMANNIAN MANIFOLDS"

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A recent article in these *Proceedings* [4] was concerned with the following theorem.

Theorem. *In order that a smooth riemannian manifold M be complete, it is necessary and sufficient that M support a smooth proper function f whose gradient is bounded in modulus.*

The purpose of this present note is to call attention to the fact that our proof of the *necessity* part of the theorem contains a serious error, and that a correct proof of this part of the theorem is contained in an article by M. P. Gaffney [3]. (See also [1] and [2] for some earlier results which were tending in this direction.)

Our proof of *necessity* used the "fact" that an isometric embedding $j: M \to R^q$ is closed if the induced metric is complete. This is easily shown to be false; e.g., the embedding of R^1 into R^2 given by $t \to (e^t, \sin t)$ is not closed but the induced metric is complete. (On the other hand, the converse statement is true; i.e. a metric induced by a closed embedding is necessarily complete.)

A correct proof can be obtained as follows: Let p be a fixed point of M and let $f(x) = d(x, p)$, where d is the distance function corresponding to the given riemannian metric. Then the completeness of M implies that f is proper. (This is merely a restatement of that portion of the Hopf-Rinow theorem which asserts that a riemannian manifold is complete if and only if closed and bounded sets are compact.) The function f is not differentiable, but it is Lipshitz continuous with Lipshitz constant = 1; i.e., $|f(x) - f(y)| \leq d(x, y)$. Using this fact, Gaffney [3] shows, by means of various smoothing operations and partition of unity constructions, that for any positive ϵ there exists a smooth

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approximation $\hat{f}$ to f which satisfies (i) $|\hat{f}(x) - f(x)| < \epsilon$ and (ii) $|\hat{f}(y) - \hat{f}(x)| < (1 + \epsilon)d(x, y)$. The first relation implies that $\hat{f}$ is proper, and the second implies that $\|\nabla \hat{f}\| \leq 1 + \epsilon$.

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