CORRECTIONS TO
"AN ANALYTICAL CRITERION FOR THE COMPLETENESS
OF RIEMANNIAN MANIFOLDS"

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A recent article in these Proceedings [4] was concerned with the follow-
ing theorem.

Theorem. In order that a smooth riemannian manifold \( M \) be complete, it is necessary and sufficient that \( M \) support a smooth proper function \( f \) whose gradient is bounded in modulus.

The purpose of this present note is to call attention to the fact that our proof of the necessity part of the theorem contains a serious error, and that a correct proof of this part of the theorem is contained in an article by M. P. Gaffney [3]. (See also [1] and [2] for some earlier results which were tending in this direction.)

Our proof of necessity used the "fact" that an isometric embedding \( j : M \to \mathbb{R}^q \) is closed if the induced metric is complete. This is easily shown to be false; e.g., the embedding of \( \mathbb{R}^1 \) into \( \mathbb{R}^2 \) given by \( t \to (e^t, \sin t) \) is not closed but the induced metric is complete. (On the other hand, the converse statement is true; i.e. a metric induced by a closed embedding is necessarily complete.)

A correct proof can be obtained as follows: Let \( p \) be a fixed point of \( M \) and let \( f(x) = d(x, p) \), where \( d \) is the distance function corresponding to the given riemannian metric. Then the completeness of \( M \) implies that \( f \) is proper. (This is merely a restatement of that portion of the Hopf-Rinow theorem which asserts that a riemannian manifold is complete if and only if closed and bounded sets are compact.) The function \( f \) is not differentiable, but it is Lipshitz continuous with Lipshitz constant = 1; i.e., \( |f(x) - f(y)| \leq d(x, y) \). Using this fact, Gaffney [3] shows, by means of various smoothing operations and partition of unity constructions, that for any positive \( \epsilon \) there exists a smooth
approximation \( \hat{f} \) to \( f \) which satisfies (i) \( |\hat{f}(x) - f(x)| < \epsilon \) and (ii) \( |\hat{f}(y) - \hat{f}(x)| < (1 + \epsilon)d(x, y) \). The first relation implies that \( \hat{f} \) is proper, and the second implies that \( \|\nabla\hat{f}\| \leq 1 + \epsilon \).

BIBLIOGRAPHY


