ON A SUBCLASS OF BAZILEVIČ FUNCTIONS

P. J. EENIGENBURG, S. S. MILLER, P. T. MOCANU AND M. O. READE

ABSTRACT. The authors show that certain Bazilevič functions are spiral-like. Then the authors study the growth and Hardy classes of those special functions.

Introduction. I. E. Bazilevič [2] gave an explicit construction for a class of functions analytic and univalent in the unit disc $D$ (see also [10]). His result was as follows.

Theorem 1. Let $g$ be univalent starlike in $D$ with $g(0) = 0$, and let $h$ be analytic and satisfy $\text{Re}(e^{i\lambda}h(z)) > 0$ in $D$ for some real $\lambda$. Then if $\alpha > 0$ and $\beta$ is real, the function

$$f(z) = \left(\int_0^z g^{\alpha}(\zeta)h(\zeta)\beta^i\zeta^{\beta-1}d\zeta\right)^{1/(\alpha+i\beta)}$$

is analytic and univalent in $D$.

In this paper we consider the functions $f$ that arise from (1) when $h(z) = 1$. Such a function $f$ must satisfy

$$\text{Re}\left[1 + z'^n(z)/f'(z) + (\alpha - 1 + i\beta)zf'(z)/f(z)\right] > 0, \quad z \in D.$$  

Conversely, if $f$ is analytic in $D$, with $f(0) = 0$, $f(z)f'(z)/z \neq 0$ ($z \in D$), and if $f$ satisfies (2) for some $\alpha > 0$, $\beta$ real, then $f$ can be written in the form (1), with $h(z) = 1$. Let us denote the class of such functions $f$ by $B(\alpha + i\beta)$. The class obtained when $\beta = 0$ has been studied extensively [5], [6], [7], [8], [9]. The class $B(1 + i\beta)$ has recently been considered by H. Yoshikawa [12]; he showed that if $f \in B(1 + i\beta)$ then $f$ is $\gamma$-spiral-like, where $\gamma = \text{arc tan } \beta$. We generalize this to

Received by the editors December 5, 1972.


Key words and phrases. Bazilevič function, univalent, Hardy class, spiral-like, growth.

1This author acknowledges support received from the National Science Foundation via NSF Grant GP 28115.
**Theorem 2.** If $f \in B(\alpha + i\beta)$ then $f$ is $\gamma$-spiral-like, where $\gamma$ satisfies $\alpha + i\beta = (\alpha^2 + \beta^2)^{1/2}e^{i\gamma}, -\pi/2 < \gamma < \pi/2$.

**Proof.** Define a function $w$ analytic in $D$ by

$$e^{i\gamma} \frac{zf'(z)}{f(z)} = \cos \gamma \frac{1 + w(z)}{1 - w(z)} + i \sin \gamma, \quad z \in D.$$  

One easily checks that $w(0) = 0$, $w(z) \neq \pm 1$ $(z \in D)$. It suffices to show $|w(z)| < 1$ for $z \in D$. Let $w(z) = R(z)e^{i\phi(z)}$ for $z = re^{i\theta}$ and suppose that $z_0 = r_0e^{i\theta_0}$ is a point of $D$ such that

$$\max_{|z| \leq r_0} |w(z)| = |w(z_0)| = 1.$$  

Then $(\partial R/\partial \theta)|_{z = z_0} = 0$, and since

$$\frac{zw'(z)}{w(z)} = \frac{\partial \phi}{\partial \theta} - i \frac{1}{R} \frac{\partial R}{\partial \theta}$$

we have at the point $z_0$,

$$z_0 w'(z_0)/w(z_0) = (\partial \phi/\partial \theta)|_{z = z_0}.$$  

We shall now show that

$$\mathbf{4)} \quad \Re \left[ 1 + zf''(z)/f'(z) + (\alpha - 1 + i\beta)zf'(z)/f(z) \right]_{z = z_0} = 0$$

thus contradicting the assertion $f \in B(\alpha + i\beta)$. By (3), (4) can be written as

$$\mathbf{5)} \quad \Re \left[ \frac{zP'(z)}{P(z) + i \tan \gamma} + (\alpha^2 + \beta^2)^{1/2}(\cos \gamma P(z) + i \sin \gamma) \right]_{z = z_0} = 0$$

where $P(z) = (1 + w(z))/(1 - w(z))$. Since $|w(z_0)| = 1$ and since $[z_0 w'(z_0)/w(z_0)]$ is real, it follows that $P(z_0)$ is imaginary and $z_0 P'(z_0)$ is real. Hence (5) holds at $z_0$. This completes the proof.

**Theorem 3.** If $\alpha' + i\beta' = t(\alpha + i\beta)$, $t \geq 1$, then $B(\alpha + i\beta) \subset B(\alpha' + i\beta')$.

**Proof.** Since $f$ is $\gamma$-spiral-like $(\alpha + i\beta = (\alpha^2 + \beta^2)^{1/2}e^{i\gamma})$,

$$\Re [(t - 1)(\alpha^2 + \beta^2)^{1/2}e^{i\gamma}zf'(z)/f(z)] \geq 0, \quad z \in D.$$  

Then

$$\Re [1 + zf''(z)/f'(z) + (\alpha' - 1 + i\beta')zf'(z)/f(z)]$$

$$= \Re [1 + zf''(z)/f'(z) + (\alpha - 1 + i\beta)zf'(z)/f(z)]$$

$$+ \Re [(t - 1)(\alpha^2 + \beta^2)^{1/2}e^{i\gamma}zf'(z)/f(z)] \geq 0, \quad z \in D.$$
In the integral representation for functions in $B(\alpha + i\beta)$, namely,

$$f(z) = \left\{ \int_0^z g^\alpha(\zeta)\zeta^{i\beta - 1} \, d\zeta \right\}^{1/(\alpha + i\beta)},$$

let us denote by $f_{\alpha+i\beta}$ the function obtained by letting $g$ be the Koebe function $z/(1-z)^2$. The following theorem illustrates the dependence of the growth of $f$ on the parameters $\alpha$ and $\beta$.

**Theorem 4.** Suppose $f \in B(\alpha + i\beta)$.

(A) If $0 < \alpha \leq \frac{1}{2}$, then, unless $f$ is a rotation or magnification of $f_{1/2+i\beta}$, $f$ is bounded.

(B) If $\alpha > \frac{1}{2}$ and $f$ is not a rotation or magnification of $f_{\alpha+i\beta}$, then there exists $\epsilon = \epsilon(f) > 0$ such that $f \in H^{\lambda+\epsilon}$ and $f' \in H^{(\lambda/2+\lambda)+\epsilon}$, where $\lambda = (\alpha^2 + \beta^2)/(2\alpha - 1)$.

(C) For $\alpha > \frac{1}{2}$ the function $f_{\alpha+i\beta}$ belongs to $H^\lambda$, $\forall \lambda < \lambda$, but does not belong to $H^\lambda$.

**Proof.** Following Sheil-Small's construction [11] of $f$ in "analytic stages" from the representation (6), we set

$$F(z) = \left(\frac{g(z)}{z}\right)^\alpha = \sum_{n=0}^\infty C_n z^n,$$

for a suitable branch of the nonvanishing function $(g(z)/z)^\alpha$. Let

$$G(z) = \sum_{n=0}^\infty \frac{C_n}{n + \alpha + i\beta} z^n,$$

so that $G$ is analytic in $D$ and satisfies the differential equation

$$(\alpha + i\beta)G(z) + zG'(z) = F(z),$$

Sheil-Small [11] shows that $G(z) \neq 0$ in $D$. We now define $f$ by the formula

$$f(z) = z[G(z)]^{1/(\alpha + i\beta)}.$$

One can easily verify that apart from a constant factor, this defines an analytic branch of the formula (6). By (9) we may write

$$G(z) = [(f(z)/z)^{1+i\beta/\alpha}]^\alpha = [s(z)/z]^{\alpha},$$

where (10) is the defining equation for $s$. Since $f$ is $\gamma$-spiral-like (Theorem 2), it follows easily that $s$ is starlike in $D$. From (8) we have

$$zG'(z) = (g(z)/z)^\alpha - (\alpha + i\beta)(s(z)/z)^\alpha.$$

If $g$ is not a rotation of the Koebe function, then there exists $\epsilon = \epsilon(g) > 0$ such that $g \in H^{1/2+\epsilon}$ [4]. Furthermore, it is easy to see from (7) that $s$ cannot be a rotation of the Koebe function. Thus, $G' \in H^{1/(2\alpha)+\epsilon}$, $\epsilon$ denoting
a positive number, not necessarily the same in each instance. Hence, if
0 < \alpha \leq \frac{1}{2}, G is bounded and so \int is bounded, by (9). Whence, (A) is proved.
For \alpha > \frac{1}{2}, a Hardy-Littlewood theorem [3, p. 88] shows that \( G \in H^{1/(2\alpha-1)} \),
hence, from (10), \( s \in H^{a/(2a-1)} \). From the identity
(12) \[ (f(z)/z)^{1+i\beta/\alpha} = s(z)/z, \]
we have
(13) \[ \left| \frac{f(z)}{z} \right| = \left| \frac{s(z)}{z} \right|^{\frac{1}{2}} \frac{\alpha^2 + \beta^2}{\alpha^2 + \beta^2} \cdot \exp \left[ \frac{\alpha \beta}{\alpha^2 + \beta^2} \arg \left( \frac{s(z)}{z} \right) \right]. \]
Since \( s \in H^{a/(2a-1)+\epsilon} \) and the exponential factor is bounded, it follows that
(14) \[ f \in H^{\lambda+\epsilon}, \quad \lambda = \frac{\alpha^2 + \beta^2}{\alpha(2\alpha - 1)}. \]

To complete the proof of (B), we must show the existence of an \( \epsilon > 0 \) such that \( f' \in H^{(1+\lambda)-1+\epsilon} \). By Theorem 2, there exists \( h, \Re(h(z)) > 0 \) in \( D \), such that
(15) \[ e^{i\gamma} z f'(z) = f(z) h(z). \]
Fix \( \epsilon \) in (14), and for small positive \( \delta \), let
(16) \[ k = k(\delta) = (\lambda + \epsilon)(1 + \lambda + \epsilon + \delta \lambda + \delta \epsilon)^{-1}. \]
Choosing \( p = (\lambda + \epsilon)k^{-1}, q = (1 + \delta)^{-1}k^{-1} \), and applying Hölder’s inequality to (15) with conjugate indices \( p \) and \( q \), it follows that
\[ \int_{-\pi}^{\pi} |f'(z)|^k d\theta \leq \left( \int_{-\pi}^{\pi} \left| \frac{f(z)}{z} \right|^k d\theta \right)^1/p \left( \int_{-\pi}^{\pi} |h(z)|^q d\theta \right)^1/q. \]
By (14) and the fact that \( kq < 1 \), we have that \( \int_{-\pi}^{\pi} |f'(z)|^k d\theta \) remains bounded as \( r \to 1 \). Hence, the proof of (B) is complete if we show the existence of \( \delta > 0 \) such that \( k = k(\delta) > \lambda(1+\lambda)^{-1} \). But this is easily checked by consideration of (16).

Finally, we leave the verification of (C) to the reader.

Remark 1. If we take \( \beta = 0 \) in Theorem 4, the result is the same as that obtained in [5].

Remark 2. Note that if we divide (2) by \( (\alpha^2 + \beta^2)^{1/2} \) and let \( \alpha + i\beta \to \infty \)
along the ray \( te^{i\gamma} \) the classes \( B(\alpha + i\beta) \) "tend" to the full class of \( \gamma \)-spiral-like functions. If we also let \( \alpha + i\beta \to \infty \) in Theorem 4, we find that \( \lambda = \lambda(\alpha, \beta) \to (2 \cos^2 \gamma)^{-1} \); and this is precisely the Hardy class result
previously known for $\gamma$-spiral-like functions [1].

Remark 3. It is interesting to observe that the level curves of $\lambda = (\alpha^2 + \beta^2)(\alpha(2\alpha - 1))^{-1}$ are the right branches of certain hyperbolas that are symmetric with respect to the real line; and they converge on the vertical line $\alpha = \frac{1}{2}$ (as $\lambda \to \infty$). For instance, if $\lambda = 1$ we obtain the right branch of the hyperbola $4(\alpha - \frac{1}{2})^2 - 4\beta^2 = 1$.

The authors wish to thank John Harrington for pointing out an error in the proof of Theorem 2.

REFERENCES