ON A SUBCLASS OF BAZILEVIĆ FUNCTIONS

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ABSTRACT. The authors show that certain Bazilević functions are spiral-like. Then the authors study the growth and Hardy classes of those special functions.

Introduction. I. E. Bazilević [2] gave an explicit construction for a class of functions analytic and univalent in the unit disc $D$ (see also [10]). His result was as follows.

Theorem 1. Let $g$ be univalent starlike in $D$ with $g(0) = 0$, and let $h$ be analytic and satisfy $\text{Re}(e^{i\lambda}h(z)) > 0$ in $D$ for some real $\lambda$. Then if $\alpha > 0$ and $\beta$ is real, the function

$$f(z) = \left\{ \int_0^z g^\alpha(\zeta)h(\zeta)\zeta^{i\beta - 1}d\zeta \right\}^{1/(\alpha + i\beta)}$$

is analytic and univalent in $D$.

In this paper we consider the functions $f$ that arise from (1) when $h(z) \equiv 1$. Such a function $f$ must satisfy

$$\text{Re}\left[ 1 + zf''(z)/f'(z) + (\alpha - 1 + i\beta)zf'(z)/f(z) \right] > 0, \quad z \in D.$$  

Conversely, if $f$ is analytic in $D$, with $f(0) = 0$, $f(z)f''(z)/f'(z) \neq 0$ ($z \in D$), and if $f$ satisfies (2) for some $\alpha > 0$, $\beta$ real, then $f$ can be written in the form (1), with $h(z) \equiv 1$. Let us denote the class of such functions $f$ by $B(\alpha + i\beta)$. The class obtained when $\beta = 0$ has been studied extensively [5], [6], [7], [8], [9]. The class $B(1 + i\beta)$ has recently been considered by H. Yoshikawa [12]; he showed that if $f \in B(1 + i\beta)$ then $f$ is $\gamma$-spiral-like, where $\gamma = \text{arc tan} \beta$. We generalize this to

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Theorem 2. If \( f \in B(\alpha + i\beta) \) then \( f \) is \( \gamma \)-spiral-like, where \( \gamma \) satisfies 
\[ \alpha + i\beta = (\alpha^2 + \beta^2)^{1/2}e^{i\gamma}, \quad -\pi/2 < \gamma < \pi/2. \]

Proof. Define a function \( w \) analytic in \( D \) by
\[
\frac{e^{i\gamma}zf'(z)}{f(z)} = \cos \gamma \left( \frac{1+w(z)}{1-w(z)} \right) + i \sin \gamma, \quad z \in D.
\]
One easily checks that \( w(0) = 0, w(z) \neq \pm 1 \ (z \in D) \). It suffices to show 
\[ |w(z)| < 1 \] for \( z \in D \). Let \( w(z) = R(z)e^{i\phi(z)} \) for \( z = re^{i\theta} \) and suppose that 
\( z_0 = r_0e^{i\theta_0} \) is a point of \( D \) such that 
\[ \max_{|z| \leq r_0} |w(z)| = |w(z_0)| = 1. \]
Then \( (\partial R/\partial \theta)|_{z=z_0} = 0 \), and since
\[
\frac{zw'(z)}{w(z)} = \frac{\partial \phi}{\partial \theta} - i \frac{1}{R} \frac{\partial R}{\partial \theta}
\]
we have at the point \( z_0 \),
\[ z_0w'(z_0)/w(z_0) = (\partial \phi/\partial \theta)|_{z=z_0}. \]
We shall now show that
\[
(4) \quad \text{Re}\left[ 1 + zf''(z)/f'(z) + (\alpha - 1 + i\beta)zf'(z)/f(z) \right]_{z=z_0} = 0
\]
thus contradicting the assertion \( f \in B(\alpha + i\beta) \). By (3), (4) can be written as
\[
(5) \quad \text{Re}\left[ \frac{zP'(z)}{P(z) + i \tan \gamma} + (\alpha^2 + \beta^2)^{1/2}(\cos \gamma P(z) + i \sin \gamma) \right]_{z=z_0} = 0
\]
where \( P(z) = (1+w(z))/(1-w(z)) \). Since \( |w(z_0)| = 1 \) and since 
\[ [z_0w'(z_0)/w(z_0)] \] is real, it follows that \( P(z_0) \) is imaginary and \( z_0P'(z_0) \) is real. Hence (5) holds at \( z_0 \). This completes the proof.

Theorem 3. If \( \alpha' + i\beta' = t(\alpha + i\beta), \ t \geq 1, \) then \( B(\alpha + i\beta) \subset B(\alpha' + i\beta') \).

Proof. Since \( f \) is \( \gamma \)-spiral-like \( (\alpha + i\beta = (\alpha^2 + \beta^2)^{1/2}e^{i\gamma}) \),
\[
\text{Re}\left[ (t-1)(\alpha^2 + \beta^2)^{1/2}e^{i\gamma}zf'(z)/f(z) \right] \geq 0, \quad z \in D.
\]
Then
\[
\text{Re}\left[ 1 + zf''(z)/f'(z) + (\alpha' - 1 + i\beta')zf'(z)/f(z) \right] = \text{Re}\left[ 1 + zf''(z)/f'(z) + (\alpha - 1 + i\beta)zf'(z)/f(z) \right]
\]
\[ + \text{Re}\left[ (t-1)(\alpha^2 + \beta^2)^{1/2}e^{i\gamma}zf'(z)/f(z) \right] \geq 0, \quad z \in D.
\]
In the integral representation for functions in $B(\alpha + i\beta)$, namely,

\[
(6) \quad f(z) = \left\{ \int_0^z g(\zeta) \zeta^{i\beta} - 1 \ d\zeta \right\}^{1/(\alpha + i\beta)},
\]

let us denote by $f_{\alpha+i\beta}$ the function obtained by letting $g$ be the Koebe function $z/(1-z)^2$. The following theorem illustrates the dependence of the growth of $f$ on the parameters $\alpha$ and $\beta$.

**Theorem 4.** Suppose $f \in B(\alpha + i\beta)$.

(A) If $0 < \alpha \leq \frac{1}{2}$, then, unless $f$ is a rotation or magnification of $f_{1/2+i\beta}$, $f$ is bounded.

(B) If $\alpha > \frac{1}{2}$ and $f$ is not a rotation or magnification of $f_{\alpha+i\beta}$, then there exists $\epsilon = \epsilon(f) > 0$ such that $f \in H^\lambda + \epsilon$ and $f' \in H(\lambda/(1+\lambda)) + \epsilon$, where $\lambda = (\alpha^2 + \beta^2)/(2\alpha - 1)$.

(C) For $\alpha > \frac{1}{2}$ the function $f_{\alpha+i\beta}$ belongs to $H^p$, $\forall p < \lambda$, but does not belong to $H^\lambda$.

**Proof.** Following Sheil-Small’s construction [11] of $f$ in “analytic stages” from the representation (6), we set

\[
F(z) = \left( \frac{g(z)}{z} \right)^\alpha = \sum_{n=0}^{\infty} C_n z^n,
\]

for a suitable branch of the nonvanishing function $(g(z)/z)^\alpha$. Let

\[
G(z) = \sum_{n=0}^{\infty} \frac{C_n}{n + \alpha + i\beta} z^n,
\]

so that $G$ is analytic in $D$ and satisfies the differential equation

\[
(\alpha + i\beta)G(z) + zG'(z) = F(z).
\]

Sheil-Small [11] shows that $G(z) \neq 0$ in $D$. We now define $f$ by the formula

\[
(9) \quad f(z) = z[G(z)]^{1/(\alpha + i\beta)}.
\]

One can easily verify that apart from a constant factor, this defines an analytic branch of the formula (6). By (9) we may write

\[
(10) \quad G(z) = [(f(z)/z)^{1+i\beta/\alpha}]^\alpha = [s(z)/z]^\alpha,
\]

where (10) is the defining equation for $s$. Since $f$ is $\gamma$-spiral-like (Theorem 2), it follows easily that $s$ is starlike in $D$. From (8) we have

\[
(11) \quad zG'(z) = (g(z)/z)^\alpha - (\alpha + i\beta)(s(z)/z)^\alpha.
\]

If $g$ is not a rotation of the Koebe function, then there exists $\epsilon = \epsilon(g) > 0$ such that $g \in H^{1/2 + \epsilon}$ [4]. Furthermore, it is easy to see from (7) that $s$ cannot be a rotation of the Koebe function. Thus, $G' \in H^{1/(2\alpha) + \epsilon}$, $\epsilon$ denoting
a positive number, not necessarily the same in each instance. Hence, if
0 < \alpha \leq \frac{1}{2}, G \text{ is bounded and so } f \text{ is bounded, by (9). Whence, (A) is proved.}

For \alpha > \frac{1}{2}, a Hardy-Littlewood theorem [3, p. 88] shows that \( G \in H^{1/(2\alpha - 1)} \epsilon \); hence, from (10), \( s \in H^{a/(2\alpha - 1)} \epsilon \). From the identity
\[
(f(z)/z)^{1+i\beta/\alpha} = s(z)/z,
\]
we have
\[
\left| \frac{f(z)}{z} \right| = \left| \frac{s(z)}{z} \right| \frac{a^2(a^2 + \beta^2)^{-1}}{\exp \left[ \frac{a\beta}{\alpha^2 + \beta^2} \arg \left( \frac{s(z)}{z} \right) \right]}.
\]
Since \( s \in H^{a/(2\alpha - 1)} \epsilon \) and the exponential factor is bounded, it follows that
\[
f \in H^{\lambda+\epsilon}, \quad \lambda = \frac{\alpha^2 + \beta^2}{\alpha(2\alpha - 1)}.
\]

To complete the proof of (B), we must show the existence of an \( \epsilon > 0 \) such that \( f' \in H^{(1+\lambda)-1} \epsilon \). By Theorem 2, there exists \( h, \mathrm{Re}(h(z)) > 0 \) in \( D \), such that
\[
e^{i\gamma} z f'(z) = f(z) h(z).
\]
Fix \( \epsilon \) in (14), and for small positive \( \delta \), let
\[
k = k(\delta) = (\lambda + \epsilon)(1 + \lambda + \epsilon + \delta \lambda + \delta \epsilon)^{-1}.
\]
Choosing \( p = (\lambda + \epsilon)k^{-1}, q = (1 + \delta)^{-1}k^{-1} \), and applying Hölder's inequality to (15) with conjugate indices \( p \) and \( q \), it follows that
\[
\int_{-\pi}^{\pi} |f'(z)|^k d\theta \leq \left( \int_{-\pi}^{\pi} \left| \frac{f(z)}{z} \right|^k d\theta \right)^{1/p} \left( \int_{-\pi}^{\pi} |h(z)|^q d\theta \right)^{1/q}.
\]
By (14) and the fact that \( kq < 1 \), we have that \( \int_{-\pi}^{\pi} |f'(z)|^k d\theta \) remains bounded as \( r \to 1 \). Hence, the proof of (B) is complete if we show the existence of \( \delta > 0 \) such that \( k = k(\delta) > \lambda(1+\lambda)^{-1} \). But this is easily checked by consideration of (16).

Finally, we leave the verification of (C) to the reader.

Remark 1. If we take \( \beta = 0 \) in Theorem 4, the result is the same as that obtained in [5].

Remark 2. Note that if we divide (2) by \( (\alpha^2 + \beta^2)^{1/2} \) and let \( \alpha + i\beta \to \infty \) along the ray \( te^{i\gamma} \) the classes \( B(\alpha + i\beta) \) "tend" to the full class of \( \gamma \)-spiral-like functions. If we also let \( \alpha + i\beta \to \infty \) in Theorem 4, we find that \( \lambda = \lambda(\alpha, \beta) \to (2 \cos^2 \gamma)^{-1} \); and this is precisely the Hardy class result.
previously known for $\gamma$-spiral-like functions [1].

Remark 3. It is interesting to observe that the level curves of $\lambda = (\alpha^2 + \beta^2)[(\alpha(2\alpha - 1))]^{-1}$ are the right branches of certain hyperbolas that are symmetric with respect to the real line; and they converge on the vertical line $\alpha = \frac{1}{2}$ (as $\lambda \to \infty$). For instance, if $\lambda = 1$ we obtain the right branch of the hyperbola $4(\alpha - \frac{1}{2})^2 - 4\beta^2 = 1$.

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