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BOUNDS FOR SOLUTIONS OF PERTURBED DIFFERENTIAL EQUATIONS

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ABSTRACT. A modified form of the Alekseev variation of constants equation is used to relate the solutions of systems of the form $\dot{x} = f(t, x, \lambda)$, λ in R^m and the perturbed system $\dot{y} = f(t, y, \psi(t)) + g(t, y)$. Hypotheses are given on the m parameter family of differential equations $\dot{x} = f(t, x, \lambda)$ so that if $\dot{\psi}$ and g are perturbation functions, bounds can be calculated for the solutions of the perturbed system.

The purpose of this paper is to study bounds for the solutions of a system of differential equations of a particular form. In the differential equation $\dot{y} = B(t, y)$ we assume B can be written as a sum $B(t, y) = f(t, y, \psi(t)) + g(t, y)$ where the m-parameter family of differential equations $\dot{x} = f(t, x, \lambda)$ has nice properties and the functions $\dot{\psi}$ and g are perturbation functions. The bounds are established using a variation of constants formula due to V. M. Alekseev [1] in a modified form in much the same way that perturbations of linear systems have been studied using the standard variation of constants formula [3, pp. 64–70]. The bounds are given as solutions of Volterra integral equations using a standard comparison method.

Let m and n be positive integers, let $c > 0$, let S_c be the closed ball of radius c in R^m and let R^+ denote the nonnegative numbers. We assume f and the matrices of derivatives f_x, f are continuous for (t, x, λ) in $R^+ \times R^n \times S_c$ ($f(t, x, \lambda)$ in R^n), ψ is in $C^1(R^+, S_c)$ and g is in $C(R^+ \times R^n, R^n)$. The solution $\phi(t, \tau, y, \lambda)$ of

$$(1) \qquad \dot{x} = f(t, x, \lambda), \qquad x(\tau) = y,$$

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for $0 \le \tau \le t$, y in R^n and λ in S_c exists for t near τ as do the matrices of derivatives ϕ_y and ϕ_λ [3].

Theorem 1. *Let $0 \le \tau < t_0$ and let $y(t)$ be a solution of*

$$\dot{y} = f(t, y, \psi(t)) + g(t, y), \qquad y(\tau) = y, \tag{2}$$

for $\tau \le t \le t_0$ where $\phi(t, s, y(s), \lambda)$ exists for $\tau \le s \le t \le t_0$. Then y satisfies

$$y(t) = \phi(t, \tau, y, \psi(\tau)) + \int_\tau^t H(t, s, y(s))\, ds, \qquad \tau \le t \le t_0, \tag{3}$$

$$H(t, s, y) = \phi_y(t, s, y, \psi(s))g(s, y) + \phi_\lambda(t, s, y, \psi(s))\dot{\psi}(s). \tag{4}$$

Proof. For $u = (x, \lambda)$ in $R^n \times S_c$ and t in R^+ the function

$$\mathcal{F}(t, u) = \text{column}\,(f(t, x), 0),$$

the matrix of derivatives $\mathcal{F}_u$ and the function

$$\mathcal{G}(t, u) = \text{column}\,(g(t, x), \dot{\psi}(t))$$

are continuous. The problem $\dot{u} = \mathcal{F}(t, u)$, $u(\tau) = \text{column}\,(y, \lambda^*) \equiv \Gamma$ with solution $\Phi(t, \tau, \Gamma) = \text{column}\,(\phi(t, \tau, y, \lambda^*), \lambda^*)$ and the problem

$$\dot{v} = \mathcal{F}(t, v) + \mathcal{G}(t, v), \qquad v(\tau) = \Gamma,$$

are related by the Alekseev formula [3, Lemma 3, p. 201]

$$v(t) = \Phi(t, \tau, \Gamma) + \int_\tau^t \Phi_\Gamma(t, s, v(s))\mathcal{G}(s, v(s))\, ds. \tag{5}$$

Equation (3) follows from (5) when $\lambda^* = \psi(\tau)$.

Remark 1. There is a corresponding version of Theorem 1 when ψ depends on t and y (see [8]); however we do not need the more complicated form for the subsequent work.

Remark 2. We recall [3, p. 22] that ϕ_λ satisfies a linear differential equation which gives the representation

$$\phi_\lambda(t, s, y, \lambda) = \int_s^t \phi_y(t, s, y, \lambda)\phi_y^{-1}(v, s, y, \lambda) f_\lambda(v, \phi(v, s, y, \lambda), \lambda)\, dv.$$

Equation (3) is a variation of constants type formula which gives a comparison between the solutions of the family of systems (1) and the perturbed system (2). We will use this equation to deduce bounds on the solutions of (3) under appropriate hypotheses. If A is an $n \times n$ matrix we denote by $\mu(A)$

its "logarithmic" norm [3]

$$\mu(A) = \lim_{h\to 0} \frac{|I + hA| - 1}{h}$$

and make the additional hypotheses:

(i) For t in R^+ and λ in S_c there are functions $\alpha(t, \lambda)$ and $\beta(t, \lambda)$ such that

$$\mu(f_x(t, x, \lambda)) \leq \alpha(t, \lambda), \qquad \mu(-f_x^T(t, x, \lambda)) \leq \beta(t, \lambda) \tag{6}$$

for (t, x, λ) in $R^+ \times R^n \times S_c$ where f_x^T denotes the transpose of f_x.

(ii) For (t, λ) in $R^+ \times S_c$, $f(t, 0, \lambda) = 0$. There is a continuous function $l(t)$ for $t \geq 0$ such that $|f_\lambda(t, x, \lambda)| \leq l(t)|x|$.

(iii) There is a continuous function h from R^+ into R^+ such that

$$|g(t, x)| \leq h(t)|x| \tag{7}$$

for (t, x) in $R^+ \times R^n$.

Remark 3. The first inequality in (6) and (ii) imply $\phi(t, s, y, \lambda)$ exists for $0 \leq s \leq t$, y in R^n, λ in S_c and

$$|\phi(t, s, y, \lambda)| \leq |y| \exp \int_s^t \alpha(\tau, \lambda)\, d\tau, \quad |\phi_y(t, s, y, \lambda)| \leq \exp \int_s^t \alpha(\tau, \lambda)\, d\tau.$$

See [2, pp. 199–200]. If $Z(t) = [\phi_y^{-1}(t, s, y, \lambda)]^T$ then $\dot{Z} = -f_x^T(t, \phi(t, s, y, \lambda), \lambda)Z$ so

$$|\phi_y^{-1}(t, s, y, \lambda)| = |Z(t)| \leq \exp \int_s^t \beta(\tau, \lambda)\, d\tau.$$

For $0 \leq s \leq t$ let $k(t, s)$ be given by

$$k(t, s) = \left[h(s) + \int_s^t l(v) \exp \int_s^v [\beta(u, \psi(s)) + \alpha(u, \psi(s))]\, du\, dv\, |\dot{\psi}(s)|\right] \cdot \exp \int_s^t \alpha(u, \psi(s))\, du.$$

Theorem 2. *Let* $\delta > 0$, *let* (i)–(iii) *be satisfied, and let* $\sigma(t)$ *be the solution of*

$$\sigma(t) = \delta \exp \int_\tau^t \alpha(s, \psi(\tau))\, d\tau + \int_\tau^t k(t, s)\sigma(s)\, ds. \tag{8}$$

If y *is a solution of* (2) *with* $|y(\tau)| \leq \delta$ *then* y *exists for* $t \geq \tau \geq 0$ *and satisfies* $|y(t)| \leq \sigma(t)$.

Proof. By Theorem 1 and Remark 3 we have if y is a solution of (2) on a maximal interval of existence then y satisfies (3) and

$$|y(t)| \leq |y(\tau)| \exp \int_\tau^t \alpha(s, \psi(\tau))\, ds + \int_\tau^t k(t, s)|y(s)|\, ds$$

on this interval. By a standard comparison theorem [5] we have y exists for $t \geq \tau$ and $|y(t)| \leq \sigma(t)$

Remark 4. The solution of (8) may be difficult to obtain exactly; however upper bounds for the solution of this linear Volterra integral equation can be obtained by means similar to those used in differential equations [6], [7], or by means of the usual method for finding the resolvant. We note that if $\dot{\psi}(s)$ is small (or zero) over much of R^+, $k(t, s)$ has approximately the form required to use Gronwall's lemma.

Remark 5. If ψ is a function of t and y, i.e. $\psi = \psi(t, y)$ (see Remark 1), or if the bound (7) is replaced by a nonlinear bound in $|y|$ then a corresponding theorem may be obtained over the interval of existence for the solution of the nonlinear analogue of (8) when the integrand in this integral equation is nondecreasing in σ.

Corollary 1. *Let* (i)–(iii) *be satisfied and assume there are positive constants* M_1 *and* M_2 *with* $M_2 < 1$ *such that*

$$\int_\tau^t \alpha(s, \psi(\tau))\, ds \leq M_1, \qquad \int_\tau^t k(t, s)\, ds \leq M_2$$

for all $t \geq \tau \geq 0$. *If* y *is a solution of* (2) *then* y *exists for* $t \geq \tau$ *and* $|y(t)| \leq |y(\tau)| e^{M_1}/(1 - M_2)$.

Proof. For $\delta = |y(\tau)|$ we obtain for the solution of (8) at any $t \geq \tau$,

$$\sigma(t) \leq |y(\tau)|\, e^{M_1} + M_2 \sup_{\tau \leq s \leq t} \sigma(s).$$

If $\sup_{\tau \leq s \leq t} \sigma(s) = \sigma(t_1)$ we have $\sigma(t_1) \leq |y(\tau)| e^{M_1}/(1 - M_2)$ which gives the result.

When the solutions $\phi(t, \tau, y, \lambda)$ are known hypothesis (i) may be replaced with bounds on ϕ and ϕ_y. The most important such case occurs when $f(t, x, \lambda) = A(\lambda)x$. We assume

(i*) There is a number M and a continuous real valued function $a(\lambda)$ for λ in S_c such that $|\exp(A(\lambda)t)| \leq M \exp(a(\lambda)t)$ for λ in S_c, $t \geq 0$.

(ii*) There is a number $N > 0$ such that

$$|(\partial a_{ij}/\partial\lambda_k)(\lambda)| \le N, \qquad i, j = 1, 2, \cdots, n, \; k = 1, 2, \cdots, m,$$

and λ in S_c. Here the $a_{ij}(\lambda)$ are the elements of $A(\lambda)$.

By (ii*) there is a number N_1 such that $|(\partial/\partial\lambda)(A(\lambda)x)| \le N_1|x|$ for λ in S_c. Thus

$$|\phi_\lambda(t, s, y, \lambda)| \le M^2 N_1(t - s)\exp(a(\lambda)(t - s))|y|.$$

For $0 \le s \le t$, let $K(t, s)$ be given by

$$K(t, s) = [h(s) + |\dot\psi(s)|M^2 N_1(t - s)]\exp(a(\psi(s))(t - s)).$$

Theorem 3. *Let* $\delta > 0$, *let* $f(t, x, \lambda) = A(\lambda)x$, g *satisfy* (i*), (ii*), (iii) *above and let* $\sigma(t)$ *be the solution of*

$$\sigma(t) = \delta \exp a(\psi(\tau))(t - \tau) + \int_\tau^t K(t, s)\sigma(s)\,ds.$$

If y *is a solution of* (2) *with* $|y(\tau)| \le \delta$, y *exists for* $t \ge \tau$ *and* $|y(t)| \le \sigma(t)$.

The proof proceeds exactly as the proof of Theorem 2. There is, of course, a corollary analogous to Corollary 1 in the case $a(\psi(\tau)) \le 0$ and $\int_\tau^t K(t, s)\,ds$ is bounded above by a number $\theta < 1$. This corollary and Corollary 1 guarantee the solution $y = 0$ of 2 is stable. Similar hypotheses would guarantee uniform stability.

By replacing the function $a(\lambda)$ with a constant, Theorem 1 and the comparison theorem for Volterra integral equations produce an improvement of a result concerning stability given in Coppel [3, p. 117].

Theorem 4. *Let* $A(t)$ *be a* C^1 *matrix defined for* $t \ge 0$ *such that for some* $\alpha > 0$, $M \ge 1$, $|\exp(A(\lambda)t)| \le Me^{-\alpha t}$ *when* $t \ge 0$, $\lambda \ge 0$. *If* $y(t)$ *is a solution of*

$$\dot y = A(t)y, \tag{9}$$

then $|y(t)| \le \sigma(t)e^{-\alpha t}$ *for* $0 \le \tau \le t$ *where* σ *is the solution of*

$$\ddot\sigma - M^2|A(t)|\sigma = 0, \quad \sigma(\tau) = Me^{\alpha\tau}|y(\tau)|, \quad \dot\sigma(\tau) = 0. \tag{10}$$

If $|\dot A(t)| \le \beta$, *and* $Y(t)$ *is a fundamental solution of* (9) *then*

$$|Y(t)Y^{-1}(\tau)| \le M\exp(-(\alpha - M\sqrt{\beta})(t - \tau)). \tag{11}$$

Proof. We take $m = 1$, $\psi(t) = t$ and $S_c = R$ in Theorem 1, then if y is a solution of (9)

$$|y(t)| \leq M\exp(-\alpha(t-\tau))|y(\tau)| + \int_\tau^t M^2(t-s)\exp(-\alpha(t-s))|A'(s)||y(s)|\,ds.$$

Hence $|y(t)|e^{\alpha t} \leq \sigma(t)$ for $t \geq \tau$, where σ satisfies

$$\sigma(t) = Me^{\alpha\tau}|y(\tau)| + \int_\tau^t M^2(t-s)|A'(s)|\sigma(s)\,ds;$$

and thus σ satisfies (10).

If $|A'(t)| \leq \beta$ then $|y(t)|e^{\alpha t} \leq \xi(t)$ for $t \geq \tau$, where ξ satisfies

$$\xi(t) = Me^{\alpha\tau}|y(\tau)| + \int_\tau^t M^2\beta(t-s)\xi(s)\,ds.$$

Thus $|y(t)| \leq M\exp(-\alpha(t-\tau))\cosh M\sqrt{\beta}(t-\tau)|y(\tau)|$ which implies (11).

Remark 6. The solution $\sigma(t)$ of (10) satisfies

$$\sigma(t) \leq (\sigma^2(t) + \dot{\sigma}^2(t))^{1/2} \leq Me^{\alpha\tau}|y(\tau)| \exp\left(\int_\tau^t \mu(s)\,ds\right),$$

where $\mu(t) = \frac{1}{2}((M^4|A'(t)|^2 + 1)^{1/2} - M^2|A'(t)|)$, by [3, Theorem 3, p. 58].

Another way to obtain bounds for the solution of (2) results from the observation that if $z(t)$ satisfies

$$(12)\qquad \begin{gathered} \dot{z} = G(t, z), \qquad z(\tau) = y, \\ G(t, z) = \phi_y^{-1}(t, \tau, z, \psi(t))[g(t, \phi(t, \tau, z, \psi(t))) - \phi_\lambda(t, \tau, z, \psi(t))\dot{\psi}(t)], \end{gathered}$$

then $y = \phi(t, \tau, z(t), \psi(t))$ satisfies (2). A similar study was given in [4] for the case when f did not depend on the parameter λ. Obvious modifications can be made to the hypotheses and conclusions of the theorems of [4] to deduce bounds for the study of (2). Again this method can be altered for the case when $\psi = \psi(t, y)$ depends on y; however the results require very complicated hypotheses.

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