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A SINGULAR PRIMITIVE RING

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ABSTRACT. An example of a primitive ring with nonzero singular ideal is constructed. An example, due to B. Osofsky, of a semiprimitive ring with nonzero singular ideal is shown to be nonprimitive.

All rings are associative with a unit element. All modules are unitary. R is a (right) primitive ring if it has a faithful irreducible right module. A right ideal of R is essential if it has nontrivial intersection with every nonzero right ideal of R. The singular ideal $Z(R)$ is the set of elements of R which annihilate essential right ideals on the left. Equivalently

$$Z(R) = \{x \in R: \forall y \ (\neq 0) \in R, \ \exists z \in R \text{ such that } yz \neq 0, \ xyz = 0\}.$$

The existence of a primitive ring with nonzero singular ideal has been an open problem for several years. In [3] Osofsky constructed an example of a semiprimitive ring with nonzero singular ideal. Both Osofsky and Faith [1, p. 128] conjectured the existence of primitive rings with singular ideal. In this paper we construct such a ring, and also show that Osofsky's ring is not primitive.

In proving the primitivity of the ring, we use several ideas from [2]. In particular we use

Theorem 1. *A ring is (right) primitive if and only if it has a proper right ideal M comaximal with every nonzero two-sided ideal of R; i.e. if J $(\neq 0)$ is a two-sided ideal, then $J + M = R$.*

The ring which we will construct is very similar to Osofsky's example in [3].

Let $F = Z_2[X, Y_j]$, $j = 1, 2, \cdots$, be the algebra over Z_2 in noncommuting variables. An arbitrary monomial in F can be written as

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$$m = X^{i_1} Y_{j_1} X^{i_2} \cdots Y_{j_{n-1}} X^{i_n}, \qquad i \geq 0,\ j \geq 1,$$

repetitions allowed. Define

$$\begin{aligned} c(m) &= \sum i_k = \text{degree of } X \text{ in } m, \\ d(m) &= \sum i_k + n - 1 = \text{degree of } m, \\ e(m) &= \max\{j_k \text{ times the number of times } Y_{j_k} \text{ appears}\}, \\ &\qquad\qquad\qquad\qquad \text{if } m \text{ contains a } Y \text{ term}, \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

Let I be the ideal of F generated by monomials m such that $c(m) > e(m) \geq 1$. Let $R = F/I$. We claim that this is the desired example. I is 'homogeneous' in the sense that a sum of distinct monomials of F is in I if and only if each monomial is in I. This allows us to speak of monomials in R. Given $(0 \neq)\ f = m_1 + \cdots + m_n$, a sum of distinct nonzero monomials in R, define

$$c(f) = \max\{c(m_i)\}, \qquad d(f) = \max\{d(m_i)\}, \qquad e(f) = \max\{e(m_i)\},$$

Theorem 2. $Z(R) \neq (0)$.

Proof. This is similar to the proof of Lemma 4 in [3]. Suppose $(0 \neq)$ $f \in R$, a sum of monomials as above. Let $s = \min\{c(m_i)\}$. Choose $t > \max\{d(f), e(f)\}$. Then $fY_t X^{t-s} \neq 0$ and $XfY_t X^{t-s} = 0$; hence $X \in Z(R)$. □

Since R is countable, we can order the nonzero elements $f_2, f_3, \cdots$. We start the numbering at 2 in order to simplify certain subsequent statements. Given f_n in this sequence, choose $j_n > 1$ large enough so that $Y_{j_n} f_n Y_{j_n} \neq 0$, and let

$$q_n = Y_n Y_1^{i_n} Y_{j_n} f_n Y_{j_n} Y_1^{i_n} \neq 0,$$

where $i_n > 2\max\{d(f_n), e(f_n)\}$. Choose $i_{n+1} > i_n$, $j_{n+1} > j_n$. Suppose $q_n = a(n, 1) + a(n, 2) + \cdots + a(n, k_n)$, a sum of distinct nonzero monomials in R. Let A be the subring of R generated by all the $a(n, m)$.

Lemma. *The following hold in A.*

1. *If m_1, m_2 are any monomials of R and $a(n, i)$, $a(m, j)$ are generators of A, then $0 \neq a(n, i)m_1 = a(m, j)m_2 \Rightarrow i = j$, $m = n$.*

2. *If $a_1, \cdots, a_n$ are generators of A, then $\prod a_i \neq 0$.*

3. *A is a free Z_2-algebra on the given generators.*

4. *If $a \in A$ and $b \in R$ and b is a sum of monomials, none of which is in A, then $ab \in A \Rightarrow ab = 0$.*

Proof. 1. Since the products are nonzero, they are equal if and only if they are identical. Since $a(n, i)$ begins with Y_n and $a(m, j)$ begins with Y_m, we conclude that $m = n$. Now

$$a(n, i)m_1 = Y_n Y_1^{i_n} Y_{j_n} b Y_{j_n} Y_1^{i_n} m_1, \qquad n \neq 1,\ j_n \neq 1.$$

As $i_n > d(b)$, we can 'decide' which generator occurs at the beginning of the product.

2. $\Pi\, a_k = \Pi\, Y_k Y_1^{i_k} Y_{j_k} b_k Y_{j_k} Y_1^{i_k}$, where $i_k > 2 \max\{d(b_k), e(b_k)\}$. If this product is zero then some segment of it must be a generator of I, say $m = Y_{j_s} b_s Y_{j_s} Y_1^{i_s} \cdots Y_t Y_1^{i_t} Y_{j_t} b_t Y_{j_t}$. If $s \neq t$, then $c(m) \leq \Sigma_{k=s}^t c(b_k)$ and $e(m) \geq \Sigma_{k=s}^t i_s$, and so $e(m) > c(m)$, a contradiction. If the generator contains only one b_k (or part of one), the result is obvious.

3. This follows from 1 and 2. If $a_{i_1} \cdots a_{i_k} = a_{j_1} \cdots a_{j_n}$, then 2 implies that this product is nonzero, and using 1 inductively, we obtain $a_{i_1} = a_{j_1}$, etc.

4. Since both I and A are generated by monomials, we will have finished if we can prove the result assuming both a and b are monomials. Suppose $a = a_{i_1} \cdots a_{i_k}$ and $0 \neq ab = a_{j_1} \cdots a_{j_n}$. Using 1, we conclude that $b \in A$, a contradiction. If $a = 1$, the result is obvious.

Theorem 3. *R is primitive.*

Proof. Let M be the right ideal of R generated by the set $\{q_n + 1\}$, $n = 2, 3, \cdots$. If $(0) \neq J$ is a two-sided ideal of R, then $f_n \in J$, for some n, hence $q_n \in J$. Then $1 \in M + J$, hence M is comaximal. If M is not proper, then for some $\{r_i\} \subset R$ and some n, we have

$$(q_2 + 1)r_2 + (q_3 + 1)r_3 + \cdots + (q_n + 1)r_n = 1.$$

Let $r_i = s_i + t_i$, where each monomial of s_i is not in A and each monomial of t_i is in A. Then $\Sigma(q_i + 1)(s_i + t_i) = 1$ and so $\Sigma(q_i + 1)s_i = 0$, by the previous lemma, and $\Sigma(q_i + 1)t_i = 1$.

The latter equation gives a nontrivial relation in the free algebra A, and hence, is impossible. Since we arrive at a contradiction by assuming that M is not proper, the theorem follows (use Theorem 1).

We conclude this paper by looking at the ring constructed in [3]. Given m, a nonzero monomial of F, define

$$e'(m) = (\max\{j_k\}) \text{ (the number of times } Y_{\max j_k} \text{ appears)}, \quad \text{if } m \text{ has a } Y \text{ term},$$

$$= 0, \text{ otherwise},$$

and define

$$e'(f) = \max\{e'(m_i)\},$$

as in the definition of $e(f)$. Let I' be the ideal of F generated by monomials m such that $c(m) > e'(m) \geq 1$, and let $R' = F/I'$. This is Osofsky's ring.

Remark. Let S be the subring of R' consisting of those elements with zero constant term. Here we waive the condition that a ring must have unity. If M is a faithful irreducible right S-module, then M is a faithful irreducible right R'-module, under the obvious action. Hence, if S is primitive, then R' is primitive.

Theorem 4 (**Osofsky** [3]). *R' is a prime semiprimitive ring and $Z(R') = (X)$.*

Theorem 5. *R' is not primitive.*

Proof. Proof by contradiction. Suppose R' is primitive. Let M be a proper right ideal of R', comaximal with every nonzero two-sided ideal. Then there exists $a \in (X)$ such that $a - 1 \in M$. Choose h large enough so that if Y_i occurs in a, then $h > i$. There exists $b \in (Y_h)$ such that $b - 1 \in M$. Let n be a positive integer and consider ba^n. Let m be any monomial in ba^n. Then $c(m) \geq n$ and $e'(m) \leq e'(b)$, by our choice of h. Thus, for sufficiently large n, $ba^n = 0$, hence

$$-1 = (b-1)a^n + (a-1)\left[\sum_{i=0}^{n-1} a^i\right] \in M,$$

a contradiction, since M was assumed to be proper.

REFERENCES

1. C. Faith, *Lectures in injective modules and quotient rings*, Lecture Notes in Math., no. 49, Springer-Verlag, Berlin and New York, 1967. MR 37 #2791.

2. E. Formanek, *Group rings of free products are primitive*, J. Algebra 26 (1973), 508–511.

3. B. Osofsky, *A non-trivial ring with non-rational injective hull*, Canad. Math. Bull. 10 (1967), 275–282. MR 35 #2930.

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