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NEW DISTORTION THEOREMS FOR FUNCTIONS OF POSITIVE REAL PART AND APPLICATIONS TO THE PARTIAL SUMS OF UNIVALENT CONVEX FUNCTIONS

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ABSTRACT. New distortion theorems are obtained for the class of functions $p(z) = 1 + c_n z^n + \cdots$ $(n \geq 1)$ which are analytic and $\operatorname{Re} p(z) > \alpha$ $(0 \leq \alpha < 1)$ in the unit disk $|z| < 1$. These are used to obtain new results regarding the partial sums of univalent convex functions.

1. **Introduction.** Let $P_{\alpha,n}$ represent the class of functions $p(z) = 1 + c_n z^n + \cdots$ $(n \geq 1)$ which are analytic and $\operatorname{Re} p(z) > \alpha$ $(0 \leq \alpha < 1)$ for $z \in E\{z : |z| < 1\}$. The class (K) consists of those normalized functions $f(z) = z + a_2 z^2 + \cdots$ which are analytic and univalent in E and have convex image domains $f(E)$, $S_n(z) = z + a_2 z^2 + \cdots + a_n z^n$ are the partial sums of $f(z)$. Theorems 1, 2 represent extensions of known distortion theorems of functions of positive real part; these are used to derive Theorems 3, 4 which represent new results concerning the partial sums of univalent convex functions.

2. **Theorems and their proofs.**

Theorem 1. *Let $p(z) \in P_{\alpha,n}$. Then for $|z| = r < 1$, and $n = 1, 2, 3, \cdots$,*

$$|zp'(z)| \leq 2nr^n \operatorname{Re}[p(z) - \alpha]/(1 - r^{2n}). \tag{1}$$

For each n and each α, equality is attained at $z = r$ for the function $p(z) = \alpha + [(1-\alpha)(1-z^n)/(1+z^n)] = 1 - 2(1-\alpha)z^n + \cdots$.

Proof. The special case of (1) when $p(z) \in P_{0,1}$ is well known. (For a simple proof see remark in [1, p. 316].) For $p(z) \in P_{0,n}$ the weaker inequality $|zp'(z)/p(z)| \leq [2nr^n/(1 - r^{2n})]$ was proven by T. H. MacGregor [4]. To prove (1) we employ a technique similar to that used in [4].

Assume $\alpha = 0$, for then the transformation $p(z) \to (p(z) - \alpha)/(1 - \alpha)$ will give the general result. If $p(z) \in P_{0,n}$ then $k(z) = (1 - p(z))/(1 + p(z)) = d_n z^n + \cdots$ is analytic for $|z| < 1$ and $|k(z)| < 1$. Hence $k(z) = z^n \phi(z)$,

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where $\phi(z)$ is analytic for $|z| < 1$ and $|\phi(z)| \leq 1$. For such functions we have [5, p. 168]

$$|\phi'(z)| \leq (1 - |\phi(z)|^2)/(1 - |z|^2) \qquad (|z| < 1). \tag{2}$$

From $z^n\phi(z) = (1 - p(z))/(1 + p(z))$ we obtain

(a) $|\phi|^2 = (1/r^{2n})|(1 - p)/(1 + p)|^2$,

(b) $|\phi'| = (1/r^{n+1})|(2zp' + n(1 - p^2))/(1 + p)^2|$,

where $r = |z|$, $p \equiv p(z)$ and $\phi \equiv \phi(z)$. Substituting (a) and (b) into (2) and then multiplying by $|1 + p|^2$ we obtain

$$|2zp' + n(1 - p^2)| \leq \frac{r^{2n}|1 + p|^2 - |1 - p|^2}{(1 - r^2)r^{n-1}},$$

$$|2zp'| \leq n|1 - p^2| + \frac{r^{2n}|1 + p|^2 - |1 - p|^2}{(1 - r^2)r^{n-1}}.$$

Thus, to prove (1) (with $\alpha = 0$), it is sufficient to show

$$n|1 - p^2| + \frac{r^{2n}|1 + p|^2 - |1 - p|^2}{(1 - r^2)r^{n-1}} \leq \frac{4nr^n \operatorname{Re} p}{1 - r^{2n}}. \tag{3}$$

Now we express $|1 + p|^2$, $|1 - p|^2$ and $\operatorname{Re} p$ in terms of $|1 - p^2|$. From $z^n\phi = (1 - p)/(1 + p)$ we obtain

(c) $|1 - p|^2 = |1 - p^2|\,|z^n\phi|$,

(d) $|1 + p|^2|z^n\phi| = |1 - p^2|$.

From (c) and (d) we have

(e) $4 \operatorname{Re} p = |1 + p|^2 - |1 - p|^2 = |1 - p^2|[(1 - |z^n\phi|^2)/|z^n\phi|]$.

Substituting (c), (d), (e) into (3) and then cancelling $|1 - p^2|$ we obtain

$$(1 - |\phi|)r^{2n-1}[-n(1 - r^2)(1 + |\phi|r^{2n}) + r(1 - r^{2n})(1 + |\phi|)] \leq 0.$$

Therefore, it is sufficient to show

$$r(1 - r^{2n})(1 + |\phi|) - n(1 - r^2)(1 + |\phi|r^{2n}) \leq 0 \qquad (0 \leq r < 1;\ |\phi| \leq 1;\ n = 1, 2, 3, \ldots). \tag{4}$$

The inequality (4) is equivalent to

$$\frac{r(1 - r^{2n})}{n(1 - r^2)} \leq \frac{1 + |\phi|r^{2n}}{1 + |\phi|} = 1 - (1 - r^{2n})\frac{|\phi|}{1 + |\phi|}. \tag{5}$$

For fixed r, the right side of (5) is minimum when $|\phi| = 1$. Hence, using $|\phi| = 1$ in (4) we obtain the sufficient condition

$$(6) \qquad n(1-r^2)(1+r^{2n}) - 2r(1-r^{2n}) \geq 0 \qquad (0 \leq r < 1;\ n = 1, 2, 3, \cdots).$$

The inequality (6) is equivalent to $(1-r^2)M(n, r) \geq 0$, where

$$M(n, r) = n + nr^{2n} - 2r\{1 + r^2 + r^4 + \cdots + r^{2(n-1)}\}.$$

We now show that $M(n, r) > 0$ for $0 \leq r < 1$ and $n \geq 1$. We have $M(1, r) = (1-r)^2 > 0$. For $n = 2, 4, 6, \cdots$ we find that

$$\begin{aligned} M(n, r) &= 2[(1-r)(1-r^{2n-1}) + (1-r^3)(1-r^{2n-3}) \\ &\qquad + (1-r^5)(1-r^{2n-5}) + \cdots + (1-r^{n-1})(1-r^{n+1})] \\ &= 2\sum_{k=1}^{n/2} (1-r^{2k-1})(1-r^{2n-2k+1}) > 0. \end{aligned}$$

For $n = 3, 5, 7, \cdots$ we find that

$$\begin{aligned} M(n, r) &= (1-r^n)^2 + 2[(1-r)(1-r^{2n-1}) + (1-r^3)(1-r^{2n-3}) \\ &\qquad + (1-r^5)(1-r^{2n-5}) + \cdots + (1-r^{n-2})(1-r^{n+2})] \\ &= (1-r^n)^2 + 2\sum_{k=1}^{(n-1)/2} (1-r^{2k-1})(1-r^{2n-2k+1}) > 0. \end{aligned}$$

This completes the proof. The statement that equality is attained in (1) at $z = r$ for the given function is easily verified. We now apply Theorem 1 to obtain an extension of a result of R. J. Libera.

Theorem 2. *Let $p(z) \in P_{\alpha,n}$. Then for $|z| = r$, $0 \leq r < 1$, $n = 1, 2, 3, \cdots$ and any complex number μ, $\operatorname{Re} \mu = \beta \geq 0$,*

$$(7) \qquad \left| \frac{zp'(z)}{p(z) - \alpha + (1-\alpha)\mu} \right| \leq \frac{2nr^n}{(1-r^n)[1 + \beta + (1-\beta)r^n]}.$$

Proof. The special case of (7) with $\alpha = 0$ and $n = 1$ was proven by R. J. Libera [3]; we employ the same technique. Assume that $\alpha = 0$, for then the transformation $p(z) \to (p(z) - \alpha)/(1-\alpha)$ will give the general result

$$\left|\frac{zp'(z)}{p(z)+\mu}\right| \leq \frac{|zp'(z)|}{\operatorname{Re} p(z)+\beta} \leq \frac{2nr^n \operatorname{Re} p(z)}{1-r^{2n}} \cdot \frac{1}{\operatorname{Re} p(z)+\beta}$$

$$\leq \frac{2nr^n}{1-r^{2n}} \cdot \frac{1}{1+\beta/|p(z)|},$$

where we have used the inequality (1) (with $\alpha = 0$). The result now follows by substituting $|p(z)| = |(1 - z^n\phi)/(1 + z^n\phi)| \leq (1 + r^n)/(1 - r^n)$.

In Theorem 2, the choice $\mu = (\alpha - c)/(1 - \alpha)$ yields

$$\left|\frac{zp'(z)}{p(z)-c}\right| \leq \frac{2n(1-\alpha)r^n}{(1-r^n)[1-\gamma+(1-2\alpha+\gamma)r^n]}$$

$$(p(z) \in P_{\alpha,n};\ 0 \leq r < 1;\ \gamma = \operatorname{Re} c \leq \alpha).$$

If we integrate the inequality (7) along the line segment joining the origin and the point z, we obtain

Corollary 1. *With the same hypotheses as in Theorem 2, we have*

$$\left|\log \frac{p(z)-\alpha+(1-\alpha)\mu}{(1-\alpha)(1+\mu)}\right| \leq \log \frac{(1+\beta)+(1-\beta)r^n}{(1+\beta)(1-r^n)}, \tag{7a}$$

$$\left|\frac{p(z)-\alpha+(1-\alpha)\mu}{(1-\alpha)(1+\mu)}\right| \leq \frac{(1+\beta)+(1-\beta)r^n}{(1+\beta)(1-r^n)}. \tag{7b}$$

In (7b), the choice $\mu = (\alpha - c)/(1 - \alpha)$ yields

$$|p(z) - c| \leq \frac{(1-\operatorname{Re} c)+(1-2\alpha+\operatorname{Re} c)r^n}{(1-\operatorname{Re} c)(1-r^n)} \qquad (\operatorname{Re} c \leq \alpha).$$

Finally, we note that in Theorem 2 the choice $\mu = \alpha/(1 - \alpha)$ yields the theorem which was recently announced by Dorothy B. Shaffer [6].

We now apply Theorem 2 to obtain a sharp result involving the partial sums of functions belonging to the class (K).

Theorem 3. *Let $f(z) = z + a_2z^2 + \cdots \in (K)$, and $S_n(z) = z + a_2z^2 + \cdots + a_nz^n$ its partial sums. Then for $n = 1, 2, 3, \cdots$ and $|z| = r < 1$,*

$$|zf'(z)/f(z) - zS_n'(z)/S_n(z)| \leq nr^n/(1-r^n). \tag{8}$$

Equality is attained, for each n, at $z = r$ for the function $f(z) = z/(1 - z)$.

Proof. It has been shown independently by this writer [2] and by T. Sheil-Small [7] that if $f(z) \in (K)$ and $S_n(z)$ are its partial sums, then $|f(z) - S_n(z)| \leq |z^n f(z)|$ and this implies $\operatorname{Re}[f(z)/S_n(z)] > \frac{1}{2}$ for $n = 1, 2, 3, \cdots$ and $|z| < 1$. Thus, since

$$p(z) = f(z)/S_n(z) = 1 + a_{n+1}z^n + \cdots \in P_{1/2,n}, \tag{9}$$

the result (8) follows readily upon substituting (9) into the inequality (7) of Theorem 2, taking $\mu = 1$ and $\alpha = \frac{1}{2}$. For the function $f(z) = z/(1 - z)$ we have

$$zf'(z)/f(z) - zS_n'(z)/S_n(z) = nz^n/(1 - z^n),$$

so that for this function equality is attained in (8) at $z = r$.

If in (7a) we take $\mu = 1$, $\alpha = \frac{1}{2}$, $p(z) = f(z)/S_n(z)$ we obtain

Corollary 2. *Let $f(z) \in (K)$, and $S_n(z)$ its partial sums. Then for $n = 1, 2, 3, \cdots$ and $|z| = r < 1$,*

$$|\log(f(z)/S_n(z))| \leq \log(1/(1 - r^n)).$$

We now apply Theorem 3 to obtain the radius of univalence and starlikeness for the partial sums of a convex function.

Theorem 4. *Let $f(z) = z + a_2z^2 + \cdots \in (K)$, and $S_n(z) = z + a_2z^2 + \cdots + a_nz^n$ its partial sums. Then $S_n(z)$ is univalent and starlike in the disk $|z| < r_0$, where r_0 is the positive root of the equation*

$$1 - (1 + n)r^n - nr^{n+1} = 0 \qquad (n \geq 2). \tag{10}$$

This result is sharp for each even $n = 2, 4, 6, \cdots$ for the function $f(z) = z/(1 - z)$.

Proof. From (8) of Theorem 3 we have

$$\operatorname{Re}(zS_n'(z)/S_n(z)) \geq \operatorname{Re}(zf'(z)/f(z)) - nr^n/(1 - r^n).$$

Since $f \in (K)$, we apply the well-known bound $\operatorname{Re}(zf'/f) \geq 1/(1 + r)$, so that

$$\operatorname{Re}\frac{zS_n'(z)}{S_n(z)} \geq \frac{1}{1 + r} - \frac{nr^n}{1 - r^n} = \frac{1 - (1 + n)r^n - nr^{n+1}}{(1 + r)(1 - r^n)} \geq 0 \tag{11}$$

if $1 - (1 + n)r^n - nr^{n+1} \geq 0$. For the function $f(z) = z/(1 - z) \in (K)$, we have

$$zS_n'(z)/S_n(z) = (1 - (1 + n)z^n + nz^{n+1})/(1 - z)(1 - z^n). \tag{12}$$

For $z = -r$ and $n = 2, 4, 6, \cdots$ the right side of (12) is identical with the right side of (11) so that for n even no improvement of (10) is possible in the class (K). We note that for each n, r_0 given by (10) satisfies $r_0 \geq \frac{1}{2}$ and, furthermore, $r_0 \to 1$ as $n \to \infty$.

REFERENCES

1. S. D. Bernardi, *The radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. **24** (1970), 312–318. MR **40** #4433.

2. ———, *Univalent convex maps of the unit disk*. I, II, III, Notices Amer. Math. Soc. **17** (1970), 441, 566; ibid. **18** (1971), 640. Abstract #70T-B54; 70T-B77; 71T-B113.

3. R. J. Libera, *Some radius of convexity problems*, Duke Math. J. **31** (1964), 143–158. MR **28** #4099.

4. T. H. MacGregor, *The radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. **14** (1963), 514–520. MR **26** #6388.

5. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952. MR **13**, 640.

6. D. B. Shaffer, *Distortion theorems for a special class of analytic functions*, Notices Amer. Math. Soc. **19** (1972), A519. Abstract #72T-B152.

7. T. Sheil-Small, *A note on the partial sums of convex schlicht functions*, Bull. London Math. Soc. **2** (1970), 165–168. MR **42** #485.

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