NEW DISTORTION THEOREMS FOR FUNCTIONS OF POSITIVE REAL PART AND APPLICATIONS TO THE PARTIAL SUMS OF UNIVALENT CONVEX FUNCTIONS

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ABSTRACT. New distortion theorems are obtained for the class of functions \( p(z) = 1 + c_nz^n + \cdots (n \geq 1) \) which are analytic and \( \text{Re} \ p(z) > \alpha \) \((0 < \alpha < 1)\) in the unit disk \( |z| < 1 \). These are used to obtain new results regarding the partial sums of univalent convex functions.

1. Introduction. Let \( P_{\alpha,n} \) represent the class of functions \( p(z) = 1 + c_nz^n + \cdots (n \geq 1) \) which are analytic and \( \text{Re} \ p(z) > \alpha \) \((0 < \alpha < 1)\) for \( z \in \mathbb{D} : |z| < 1 \). The class \( (K) \) consists of those normalized functions \( f(z) = z + a_2z^2 + \cdots \) which are analytic and univalent in \( \mathbb{D} \) and have convex image domains \( f(\mathbb{D}) = z + a_2z^2 + \cdots + a_nz^n \) are the partial sums of \( f(z) \). Theorems 1, 2 represent extensions of known distortion theorems of functions of positive real part; these are used to derive Theorems 3, 4 which represent new results concerning the partial sums of univalent convex functions.

2. Theorems and their proofs.

Theorem 1. Let \( p(z) \in P_{\alpha,n} \). Then for \( |z| = r < 1 \), and \( n = 1, 2, 3, \ldots \),

\[
|zp'(z)| \leq 2nr^n \text{Re}[p(z) - \alpha]/(1 - r^{2n}).
\]

For each \( n \) and each \( \alpha \), equality is attained at \( z = r \) for the function \( p(z) = \alpha + [(1 - \alpha)(1 - z^n)/(1 + z^n)] = 1 - 2(1 - \alpha)z^n + \cdots \).

Proof. The special case of (1) when \( p(z) \in P_{0,1} \) is well known. (For a simple proof see remark in [1, p. 316].) For \( p(z) \in P_{0,n} \) the weaker inequality \( |zp'(z)/p(z)| \leq [2nr^n/(1 - r^{2n})] \) was proven by T. H. MacGregor [4]. To prove (1) we employ a technique similar to that used in [4].

Assume \( \alpha = 0 \), for then the transformation \( p(z) \rightarrow (p(z) - \alpha)/(1 - \alpha) \) will give the general result. If \( p(z) \in P_{0,n} \) then \( k(z) = (1 - p(z))/(1 + p(z)) = d_nz^n + \cdots \) is analytic for \( |z| < 1 \) and \( |k(z)| < 1 \). Hence \( k(z) = z^n\phi(z) \),

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where \( \phi(z) \) is analytic for \( |z| < 1 \) and \( |\phi(z)| \leq 1 \). For such functions we have [5, p. 168]

\[
|\phi'(z)| \leq \frac{(1 - |\phi(z)|^2)/(1 - |z|^2)}{(1 - |z|^2)} \quad (|z| < 1),
\]

From \( z^n\phi(z) = (1 - p(z))/(1 + p(z)) \) we obtain

(a) \( |\phi|^2 = (1/r^{2n})|(1 - p)/(1 + p)|^2 \),
(b) \( |\phi'| = (1/r^{n+1})|(2zp' + n(1 - p^2))/(1 + p)^2| \),

where \( r = |z| \), \( p \equiv p(z) \) and \( \phi \equiv \phi(z) \). Substituting (a) and (b) into (2) and then multiplying by \( |1 + p|^2 \) we obtain

\[
|2zp' + n(1 - p^2)| \leq \frac{r^{2n}|1 + p|^2 - |1 - p|^2}{(1 - r^2)r^{n-1}},
\]

Thus, to prove (1) (with \( \alpha = 0 \)), it is sufficient to show

\[
\frac{n|1 - p|^2 + r^{2n}|1 + p|^2 - |1 - p|^2}{(1 - r^2)r^{n-1}} \leq \frac{4nr^n \text{Re} p}{1 - r^{2n}}.
\]

Now we express \( |1 + p|^2 \), \( |1 - p|^2 \) and \( \text{Re} p \) in terms of \( |1 - p^2| \). From \( z^n\phi = (1 - p)/(1 + p) \) we obtain

(c) \( |1 - p|^2 = |1 - p^2||z^n\phi| \),
(d) \( |1 + p|^2|z^n\phi| = |1 - p^2| \).

From (c) and (d) we have

(e) \( 4 \text{Re} p = |1 + p|^2 - |1 - p|^2 = |1 - p^2|[(1 - |z^n\phi|^2)/|z^n\phi|] \).

Substituting (c), (d), (e) into (3) and then cancelling \( |1 - p^2| \) we obtain

\[
(1 - |\phi|)r^{2n-1}[n(1 - r^2)(1 + |\phi|r^{2n}) + r(1 - r^{2n})(1 + |\phi|)] \leq 0.
\]

Therefore, it is sufficient to show

\[
\frac{r(1 - r^{2n})(1 + |\phi|) - n(1 - r^2)(1 + |\phi|r^{2n})}{n(1 - r^2)(1 + |\phi|r^{2n})} \leq 0 \quad (0 \leq r < 1; \ |\phi| \leq 1; \ n = 1, 2, 3, \ldots).
\]

The inequality (4) is equivalent to

\[
\frac{n(1 - r^{2n})}{n(1 - r^{2n})} \leq \frac{1 + |\phi|r^{2n}}{1 + |\phi|} = 1 - (1 - r^{2n}) \frac{|\phi|}{1 + |\phi|}.
\]
For fixed \( r \), the right side of (5) is minimum when \( |\phi'| = 1 \). Hence, using \( |\phi'| = 1 \) in (4) we obtain the sufficient condition

\[
(6) \quad n(1 - r^2)(1 + r^{2n}) - 2r(1 - r^{2n}) \geq 0 \quad (0 \leq r < 1; \ n = 1, 2, 3, \ldots).
\]

The inequality (6) is equivalent to \((1 - r^2)M(n, r) \geq 0\), where

\[
M(n, r) = n + nr^{2n} - 2r(1 + r^2 + r^4 + \cdots + r^{2(n-1)}).
\]

We now show that \( M(n, r) > 0 \) for \( 0 < r < 1 \) and \( r > 1 \). We have \( M(1, r) = (1 - r^2)^2 > 0 \). For \( n = 2, 4, 6, \ldots \) we find that

\[
M(n, r) = 2((1 - r)(1 - r^{2n-1}) + (1 - r^3)(1 - r^{2n-3}) + (1 - r^5)(1 - r^{2n-5}) + \cdots + (1 - r^{n-1})(1 - r^{n+1}))
\]

\[
= 2 \sum_{k=1}^{n/2} (1 - r^{2k-1})(1 - r^{2n-2k+1}) > 0.
\]

For \( n = 3, 5, 7, \ldots \) we find that

\[
M(n, r) = (1 - r^n)^2 + 2((1 - r)(1 - r^{2n-1}) + (1 - r^3)(1 - r^{2n-3}) + (1 - r^5)(1 - r^{2n-5}) + \cdots + (1 - r^{n-2})(1 - r^{n+2}))
\]

\[
= (1 - r^n)^2 + 2 \sum_{k=1}^{(n-1)/2} (1 - r^{2k-1})(1 - r^{2n-2k+1}) > 0.
\]

This completes the proof. The statement that equality is attained in (1) at \( z = r \) for the given function is easily verified. We now apply Theorem 1 to obtain an extension of a result of R. J. Libera.

**Theorem 2.** Let \( p(z) \in P_{a,n} \). Then for \( |z| = r, 0 \leq r < 1, n = 1, 2, 3, \ldots \) and any complex number \( \mu, \text{Re} \mu = \beta \geq 0 \),

\[
(7) \quad \left| \frac{zp'(z)}{p(z) - \alpha + (1 - \alpha)\mu} \right| \leq \frac{2nr^n}{(1 - r^n)[1 + \beta + (1 - \beta)r^n]}.
\]

**Proof.** The special case of (7) with \( \alpha = 0 \) and \( n = 1 \) was proven by R. J. Libera [3]; we employ the same technique. Assume that \( \alpha = 0 \), for then the transformation \( p(z) \rightarrow (p(z) - \alpha)/(1 - \alpha) \) will give the general result.
\[
\left| \frac{zp'(z)}{p(z) + \mu} \right| \leq \frac{|zp'(z)|}{Re \, p(z) + \beta} \leq \frac{2nr^n \, Re \, p(z)}{1 - r^{2n}} \cdot \frac{1}{Re \, p(z) + \beta} \\
\leq \frac{2nr^n}{1 - r^{2n}} \cdot \frac{1}{1 + \beta/|p(z)|},
\]

where we have used the inequality (1) (with \( a = 0 \)). The result now follows by substituting \(|p(z)| = |(1 - z^n\phi)/(1 + z^n\phi)| \leq (1 + r^n)/(1 - r^n)\).

In Theorem 2, the choice \( p = (a - c)/(1 - a) \) yields

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2n(1 - \alpha)^n}{(1 - r^n)[1 - \gamma + (1 - 2\alpha + \gamma)r^n]} \\
(p(z) \in P_{\alpha,n}; \ 0 \leq r < 1; \ \gamma = Re \ c \leq \alpha).
\]

If we integrate the inequality (7) along the line segment joining the origin and the point \( z \), we obtain

**Corollary 1.** With the same hypotheses as in Theorem 2, we have

\[
\begin{align*}
(7a) & \quad \left| \log \frac{p(z) - \alpha + (1 - \alpha)\mu}{(1 - \alpha)(1 + \mu)} \right| \leq \log \frac{(1 + \beta) + (1 - \beta)r^n}{(1 + \beta)(1 - r^n)}, \\
(7b) & \quad \left| \frac{p(z) - \alpha + (1 - \alpha)\mu}{(1 - \alpha)(1 + \mu)} \right| \leq \frac{(1 + \beta) + (1 - \beta)r^n}{(1 + \beta)(1 - r^n)}. 
\end{align*}
\]

In (7b), the choice \( p = (a - c)/(1 - a) \) yields

\[
|p(z) - c| \leq \frac{(1 - Re \ c) + (1 - 2\alpha + Re \ c)r^n}{(1 - Re \ c)(1 - r^n)} \quad (Re \ c \leq \alpha).
\]

Finally, we note that in Theorem 2 the choice \( p = a/(1 - a) \) yields the theorem which was recently announced by Dorothy B. Shaffer [6].

We now apply Theorem 2 to obtain a sharp result involving the partial sums of functions belonging to the class \((K)\).

**Theorem 3.** Let \( f(z) = z + a_2z^2 + \cdots \in (K) \), and \( S_n(z) = z + a_2z^2 + \cdots + a_nz^n \) its partial sums. Then for \( n = 1, 2, 3, \ldots \) and \(|z| = r < 1\),

\[
|z|f'(z)/f(z) - zS'_n(z)/S_n(z)| \leq nr^n/(1 - r^n).
\]

Equality is attained, for each \( n \), at \( z = r \) for the function \( f(z) = z/(1 - z) \).
Proof. It has been shown independently by this writer [2] and by T. Sheil-Small [7] that if \( f(z) \in (K) \) and \( S_n(z) \) are its partial sums, then \( |f(z) - S_n(z)| \leq |z^n f(z)| \) and this implies \( \text{Re} \left[ \frac{f(z)}{S_n(z)} \right] > \frac{1}{2} \) for \( n = 1, 2, 3, \ldots \) and \( |z| < 1 \). Thus, since
\[
(9) \quad p(z) = \frac{f(z)}{S_n(z)} = 1 + a_{n+1} z^n + \cdots \in P_{1/2,n},
\]
the result (8) follows readily upon substituting (9) into the inequality (7) of Theorem 2, taking \( \mu = 1 \) and \( \alpha = \frac{1}{2} \). For the function \( f(z) = z/(1 - z) \) we have
\[
z f'(z)/f(z) - nz^n/(1 - z^n),
\]
so that for this function equality is attained in (8) at \( z = r \).

If in (7a) we take \( \mu = 1, \alpha = \frac{1}{2}, p(z) = f(z)/S_n(z) \) we obtain

**Corollary 2.** Let \( f(z) \in (K) \), and \( S_n(z) \) its partial sums. Then for \( n = 1, 2, 3, \ldots \) and \( |z| = r < 1 \),
\[
|\log\left( \frac{f(z)}{S_n(z)} \right)| \leq \log\left( \frac{1}{1 - r^n} \right).
\]

We now apply Theorem 3 to obtain the radius of univalence and starlikeness for the partial sums of a convex function.

**Theorem 4.** Let \( f(z) = z + a_2 z^2 + \cdots \in (K) \), and \( S_n(z) = z + a_2 z^2 + \cdots + a_n z^n \) its partial sums. Then \( S_n(z) \) is univalent and starlike in the disk \( |z| < r_0 \), where \( r_0 \) is the positive root of the equation
\[
1 - (1 + n)r^n - nr^{n+1} = 0 \quad (n \geq 2).
\]
This result is sharp for each even \( n = 2, 4, 6, \cdots \) for the function \( f(z) = z/(1 - z) \).

**Proof.** From (8) of Theorem 3 we have
\[
\text{Re} \left( \frac{z S_n'(z)}{S_n(z)} \right) \geq \text{Re} \left( \frac{z f'(z)}{f(z)} \right) - nr^n/(1 - r^n).
\]
Since \( f \in (K) \), we apply the well-known bound \( \text{Re} \left( \frac{z f'/f}{1 + r} \right) \geq 1/(1 + r) \), so that
\[
(11) \quad \frac{z S_n'(z)}{S_n(z)} \geq \frac{1}{1 + r} - \frac{nr^n}{1 - r^n} = \frac{1 - (1 + n)r^n - nr^{n+1}}{(1 + r)(1 - r^n)} \geq 0
\]
if \( 1 - (1 + n)r^n - nr^{n+1} \geq 0 \). For the function \( f(z) = z/(1 - z) \in (K) \), we have
\[
z S_n'(z)/S_n(z) = (1 - (1 + n)z^n + nz^{n+1})/(1 - z)(1 - z^n),
\]
For $z = -r$ and $n = 2, 4, 6, \ldots$ the right side of (12) is identical with the right side of (11) so that for $n$ even no improvement of (10) is possible in the class $(K)$. We note that for each $n$, $r_0$ given by (10) satisfies $r_0 \geq \frac{1}{2}$ and, furthermore, $r_0 \to 1$ as $n \to \infty$.

REFERENCES


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