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TWO NONEQUIVALENT CONDITIONS FOR WEIGHT FUNCTIONS

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ABSTRACT. A nonnegative function on the real line satisfies the condition A_∞ if, given $\epsilon > 0$, there exists a $\delta > 0$ such that if I is an interval, $E \subset I$, and $|E| < \delta|I|$, then $\int_E W \le \epsilon \int_I W$. A nonnegative function on the real line satisfies the condition A if for every interval I, $\int_{2I} W \le C \int_I W$, where $2I$ is the interval with the same center as I and twice as long, and C is independent of I. An example is given of a function that satisfies A but not A_∞.

1. **Introduction.** The condition A_∞ is important in [1], [3] and [5] since for various pairs of operators, T and U, there is a constant C such that $\int |Tf|^p W \le C \int |Uf|^p W$ provided W satisfies A_∞. In [6] an inequality of this type is proved for W satisfying A. The proof in [6] would be much simpler if W were assumed to satisfy A_∞. For this reason and the fact that both A_∞ and A appear quite naturally in weighted norm inequality problems, it seems to be of interest to give an example of a function that satisfies A but not A_∞.

An example of such a function in a dyadic setting was discovered by D. Burkholder and communicated to us orally. The dyadic example provided the inspiration for parts of this paper.

If W satisfies the condition A_∞, it follows from the results of [4] or the alternate definition of A_∞ used in [6] that W satisfies A. It is also easy to show this directly as follows. Suppose that W satisfies A_∞, and let δ correspond to $\epsilon = \frac{1}{2}$ in the definition of A_∞. Let n be the least integer greater than $1/\delta$, and, given an interval I, let I_k be the interval with the same center as I and with $|I_k| = (n+k)|I|/n$. By the definition of δ

$$\int_{I_{k+1}-I_k} W \le \frac{1}{2} \int_{I_{k+1}} W.$$

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Therefore,

$$\int_{I_{k+1}} W \leq 2 \int_{I_k} W,$$

and

$$\int_{2I} W = \int_{I_n} W \leq 2^n \int_{I_0} W = 2^n \int_I W.$$

The example of this paper then shows that A_∞ is a condition that is strictly stronger than A. The example, $W(x)$, developed here is a one dimensional one, but an n dimensional example can be obtained by defining $W_n(x) = W(x_1)$, where x_1 is the first coordinate of x. The fact that W_n satisfies A but not A_∞ in R^n follows immediately from the fact that W has this property in R.

Throughout this paper the letters i, j, k, m and n will denote integers whether this is stated in the context or not; the letters a, b, h and x will be arbitrary real numbers, and C will denote a positive constant not necessarily the same at each occurrence.

2. **Definitions and simple observations.** Define

$$f_n(x) = \begin{cases} 1/2, & 0 < x \leq 4^{-n-1}, \\ 3/2, & 4^{-n-1} < x \leq 3 \cdot 4^{-n-1}, \\ 1/2, & 3 \cdot 4^{n-1} < x \leq 4^{-n}, \end{cases} \tag{2.1}$$

and to have period 4^{-n}. An equivalent definition is

$$f_n(x) = 1 - (r_{2n}(x) r_{2n+1}(x))/2, \tag{2.2}$$

where $r_n(x) = \operatorname{sgn}[\sin(2^{n+1}\pi x)]$ is the nth Rademacher function. For positive x define $n(x)$ to be the integer such that

$$4^{-n(x)-1} < x \leq 4^{-n(x)}, \tag{2.3}$$

and define

$$W(x) = \prod_{k=0}^{2n(x)} f_k(x), \qquad 0 < x \leq 1. \tag{2.4}$$

It will be shown that $W(x)$ satisfies condition A but not A_∞ on $[0, 1]$. If $W(x)$ is defined to be 1 for $x > 1$ and if $W(x)$ is defined to be $W(-x)$ for negative x, it follows trivially from the result on $[0, 1]$ that $W(x)$ satisfies the condition A but not A_∞ on $(-\infty, \infty)$.

Next define

$$W_h(x) = \prod_{k=0}^{m(x,h)} f_k(x), \tag{2.5}$$

where $m(x, h)$ is the smaller of $2n(x)$ and $n(|h|)$. The following elementary observations will be needed; proofs are given after the statements.

1. $\int_a^{a+4^{-j}} \Pi_{k=m}^n f_k(x)\,dx = 4^{-j}$ provided $j \le m \le n$.

2. $\Pi_{k=0}^n f_k(x)$ is constant on the interval $(m4^{-n-1}, (m+1)4^{-n-1}]$ for m an integer and on adjacent intervals of this type differs by a factor of 3, 1/3 or 1.

3. On $(\frac{1}{2}4^{-n}, 4^{-n}]$, $W(x) = 3^k 2^{-2n-1}$ on a subset of measure $\binom{n+1}{k} 2^{-3n-2}$.

4. If $0 < x < 1$, $-x/2 \le h \le x$ and $0 \le \theta \le 1$, then

$$W_h(x)/324 \le W_h(x + \theta h) \le 729 W_h(x)/4.$$

Because of periodicity it is sufficient to prove observation 1 for the case $a = 0$. On an interval $I_i = (i4^{-n}, (i+1)4^{-n}]$, $\Pi_{k=m}^{n-1} f_k(x)$ is constant, and the integral of $f_n(x)$ is 4^{-n}. Therefore,

$$\int_{I_i} \prod_{k=m}^{n} f_k(x)\,dx = \int_{I_i} \prod_{k=m}^{n-1} f_k(x)\,dx.$$

Adding these for $0 \le i < 4^{n-j}$ shows that

$$\int_0^{4^{-j}} \prod_{k=m}^{n} f_k(x)\,dx = \int_0^{4^{-j}} \prod_{k=m}^{n-1} f_k(x)\,dx.$$

This process then can be repeated to obtain the result.

Observation 2 follows from the fact that $f_k(x)$ changes by a factor of 3 at the points $(2j+1)4^{-k-1} = (2j+1)4^{n-k}4^{-n-1}$ and these points cannot coincide for different k's.

For observation 3, first note that $f_k(x) = \frac{1}{2}$ for x in $(\frac{1}{2}4^{-n}, 4^{-n}]$ and $0 \le k \le n-1$. The functions $f_n, \cdots, f_{2n}$ are independently distributed on $(\frac{1}{2}4^{-n}, 4^{-n}]$ since each has a full period where the preceding one is constant; this independence also follows immediately from (2.2). Furthermore, each of these functions equals $\frac{1}{2}$ on half the interval and $3/2$ on the other half. Therefore, on $(\frac{1}{2}4^{-n}, 4^{-n}]$, $\prod_{k=n}^{2n} f_k(x)$ equals $3^k 2^{-n-1}$ on a subset of measure $\binom{n+1}{k} 2^{-3n-2}$. Observation 3 follows immediately from this.

To prove observation 4, fix x and h satisfying the conditions and let $g(y) = \prod_{k=0}^{m(x,h)} f_k(y)$. By observation 2, $g(y)$ changes by at most a factor of 3 on an interval of length $4^{-n(|h|)-1}$. Since $|\theta h| \le |h| \le 4^{-n(|h|)}$, $3^{-4} g(x) \le g(x+\theta h) \le 3^4 g(x)$. Since $-x/2 \le h \le x$, the product for $W_h(x+\theta h)$ can have at most two more or two fewer f_k's in it than $g(x+\theta h)$ so

$$2^{-2} g(x+\theta h) \le W_h(x+\theta h) \le 3^2 2^{-2} g(x+\theta h).$$

Combining these inequalities and using the fact that $g(x) = W_h(x)$ proves observation 4.

3. W **satisfies condition A on** $[0, 1]$. It will be shown that there are positive constants, C_1 and C_2, such that if $0 \le h \le b$ and $b + h \le 1$, then

$$C_1 h W_h(b) \le \int_b^{b+h} W(x)\,dx \le C_2 h W_h(b), \tag{3.1}$$

and

$$C_1 h W_h(b) \le \int_{b-h}^{b} W(x)\,dx \le C_2 h W_h(b). \tag{3.2}$$

These show that the integrals of W over two adjacent subintervals of $[0, 1]$ of the same size are comparable. This obviously implies condition A by considering the two halves of the interval that is to be doubled.

To prove the second inequality in (3.1) write the integral as

$$\int_b^{b+h} W_h(x) \frac{W(x)}{W_h(x)}\,dx. \tag{3.3}$$

By observation 4, (3.3) is bounded by

$$C W_h(b) \int_b^{b+h} \frac{W(x)}{W_h(x)}\,dx; \tag{3.4}$$

as noted in §1, C will denote a positive constant not necessarily the same at each occurrence. By (2.3), (2.4) and (2.5), (3.4) is bounded by

$$C W_h(b) \int_b^{b+4^{-n(h)}} \prod_{k=m(b,h)+1}^{2n(h)} f_k(x)\,dx. \tag{3.5}$$

Then observation 1 and the fact that $4^{-n(h)} \le 4h$ complete the proof of the right inequality in (3.1). For the left inequality in (3.1), proceed in the same manner, and use the first inequality in observation 4 to show that (3.3) is bounded below by (3.4) with a different positive C. Next, (3.4) is bounded below by

$$CW_h(b) \int_b^{b+4^{-n(h)-1}} \prod_{k=m(b,h)+1}^{2n(b)} f_k(x)\, dx. \tag{3.6}$$

Observation 1 and the fact that $4^{-n(h)-1} \ge h/4$ complete this part.

If $0 < h \le b/2$, the inequalities (3.2) follow immediately from (3.1) and property 4. If $b/2 < h \le b$, then by the case just observed

$$\int_{b-h}^{b} W(x)\, dx \ge \int_{b/2}^{b} W(x)\, dx \ge C_1 \frac{b}{2} W_{b/2}(b).$$

By the definition of W_h this is bounded below by $ChW_h(b)$, and the first inequality in (3.2) is proved in this case also. Finally, to prove the second inequality in (3.2) if $b/2 < h \le b$, use the fact that

$$\int_{b-h}^{b} W(x)\, dx \le \int_0^{4^{-n(b)}} W(x)\, dx = \sum_{j=n(b)}^{\infty} \int_{4^{-j-1}}^{4^{-j}} W(x)\, dx. \tag{3.7}$$

Now on $[4^{-j-1}, 4^{-j}]$, $f_k(x) = ½$ for $0 \le k \le j-1$ and $f_j(x) \le 3/2$. Therefore, by observation 1,

$$\int_{4^{-j-1}}^{4^{-j}} W(x)\, dx \le 3 \cdot 2^{-j} \int_{4^{-j-1}}^{4^{-j}} \prod_{k=j+1}^{2j} f_k(x)\, dx = 9 \cdot 2^{-3j-2},$$

and by (3.7)

$$\int_{b-h}^{b} W(x)\, dx \le C \cdot 2^{-3n(b)}. \tag{3.8}$$

Since $b/2 < h \le b$, $4^{-n(b)-2} < h \le 4^{-n(b)}$, and $n(h)$ equals $n(b)$ or $n(b)+1$. Therefore, $2^{-2n(b)} \le 4 \cdot 2^{-2n(h)}$, and the definitions of W_h and f_k show that $2^{-n(b)} \le 2 \prod_{k=0}^{n(b)} f_k(b) \le 4W_h(b)$. Writing the right side of (3.8) as $C \cdot 2^{-n(b)} 2^{-2n(b)}$ and using these two facts complete the proof of the right inequality in (3.2)

4. W **does not satisfy** A_∞ **on** $[0, 1]$. This will be shown by producing a sequence of intervals, I_n, and a sequence of sets, E_n, such that

$$E_n \subset I_n, \qquad \lim_{n\to\infty} \frac{|E_n|}{|I_n|} = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_{E_N} W \Big/ \int_{I_n} W = 1.$$

To do this let $I_n = [\tfrac{1}{2}4^{-n}, 4^{-n}]$ and let E_n be the subset of I_n where $W(x) > 3^{2n/3}2^{-2n-1}$. By observation 3,

$$\frac{|E_n|}{|I_n|} = \sum_{k=[2n/3]+1}^{n+1} \binom{n+1}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n+1-k},$$

and

$$\int_{E_n} W \Big/ \int_{I_n} W = \sum_{k=[2n/3]+1}^{n+1} \binom{n+1}{k}\left(\frac{3}{4}\right)^k\left(\frac{1}{4}\right)^{n+1-k},$$

where $[2n/3]$ denotes the greatest integer less than or equal to $2n/3$. That the limits of these two ratios are 0 and 1 respectively follows immediately from the fact [2, (16.4.6), p. 200] that if $0 < p < 1$, $q = 1 - p$ and $a < b$, then

$$\lim_{n\to\infty} \sum_{np+a(npq)^{1/2}<k\le np+b(npq)^{1/2}} \binom{n}{k} p^k q^{n-k} = (2\pi)^{-1/2} \int_a^b e^{-x^2/2}\,dx.$$

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