TWO NONEQUIVALENT CONDITIONS FOR WEIGHT FUNCTIONS

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ABSTRACT. A nonnegative function on the real line satisfies the condition $A_\infty$ if, given $\varepsilon > 0$, there exists a $\delta > 0$ such that if $I$ is an interval, $E \subset I$, and $|E| < \delta |I|$, then $\int_E W \leq \varepsilon \int_I W$. A nonnegative function on the real line satisfies the condition $A$ if for every interval $I$, $\int_{2I} W \leq C \int_I W$, where $2I$ is the interval with the same center as $I$ and twice as long, and $C$ is independent of $I$. An example is given of a function that satisfies $A$ but not $A_\infty$.

1. Introduction. The condition $A_\infty$ is important in [1], [3] and [5] since for various pairs of operators, $T$ and $U$, there is a constant $C$ such that $\int |Tf|^p W \leq C \int |Uf|^p W$ provided $W$ satisfies $A_\infty$. In [6] an inequality of this type is proved for $W$ satisfying $A$. The proof in [6] would be much simpler if $W$ were assumed to satisfy $A_\infty$. For this reason and the fact that both $A_\infty$ and $A$ appear quite naturally in weighted norm inequality problems, it seems to be of interest to give an example of a function that satisfies $A$ but not $A_\infty$.

An example of such a function in a dyadic setting was discovered by D. Burkholder and communicated to us orally. The dyadic example provided the inspiration for parts of this paper.

If $W$ satisfies the condition $A_\infty$, it follows from the results of [4] or the alternate definition of $A_\infty$ used in [6] that $W$ satisfies $A$. It is also easy to show this directly as follows. Suppose that $W$ satisfies $A_\infty$, and let $\delta$ correspond to $\varepsilon = \frac{1}{2}$ in the definition of $A_\infty$. Let $n$ be the least integer greater than $1/\delta$, and, given an interval $I$, let $I_k$ be the interval with the same center as $I$ and with $|I_k| = (n + k)|I|/n$. By the definition of $\delta$

$$\int_{I_{k+1}} W \leq \frac{1}{2} \int_{I_{k+1}} W,$$
Therefore,
\[ \int_{I_{k+1}} W \leq 2 \int_{I_k} W, \]
and
\[ \int_{2I} W = \int_{I_n} W \leq 2^n \int_{I_0} W = 2^n \int_{I} W. \]

The example of this paper then shows that \( A_{\infty} \) is a condition that is strictly stronger than \( A \). The example, \( W(x) \), developed here is a one dimensional one, but an \( n \) dimensional example can be obtained by defining \( W_n(x) = W(x_1) \), where \( x_1 \) is the first coordinate of \( x \). The fact that \( W_n \) satisfies \( A \) but not \( A_{\infty} \) in \( R^n \) follows immediately from the fact that \( W \) has this property in \( R \).

Throughout this paper the letters \( i, j, k, m \) and \( n \) will denote integers whether this is stated in the context or not; the letters \( a, b, h \) and \( x \) will be arbitrary real numbers, and \( C \) will denote a positive constant not necessarily the same at each occurrence.

2. Definitions and simple observations. Define

\[ f_n(x) = \begin{cases} 
1/2, & 0 < x \leq 4^{-n-1}, \\
3/2, & 4^{-n-1} < x \leq 3 \cdot 4^{-n-1}, \\
1/2, & 3 \cdot 4^{-n-1} < x \leq 4^{-n},
\end{cases} \]

and to have period \( 4^{-n} \). An equivalent definition is

\[ f_n(x) = 1 - (r_{2n}(x)r_{2n+1}(x))/2, \]

where \( r_n(x) = \text{sgn}[\sin(2^{n+1}\pi x)] \) is the \( n \)th Rademacher function. For positive \( x \) define \( n(x) \) to be the integer such that

\[ 4^{-n(x)-1} < x \leq 4^{-n(x)}, \]

and define

\[ W(x) = \prod_{k=0}^{2n(x)} f_k(x), \quad 0 < x \leq 1. \]

It will be shown that \( W(x) \) satisfies condition \( A \) but not \( A_{\infty} \) on \([0, 1]\). If \( W(x) \) is defined to be 1 for \( x > 1 \) and if \( W(x) \) is defined to be \( W(-x) \) for
negative $x$, it follows trivially from the result on $[0, 1]$ that $W(x)$ satisfies
the condition $A$ but not $A_{\infty}$ on $(-\infty, \infty)$.

Next define

$$W_b(x) = \prod_{k=0}^{m(x, h)} f_k(x),$$

where $m(x, h)$ is the smaller of $2n(x)$ and $n(|h|)$. The following elementary
observations will be needed; proofs are given after the statements.

1. $\int_0^{a+4^{-j}} \prod_{k=m}^{n} f_k(x) dx = 4^{-j}$ provided $j \leq m \leq n$.

2. $\prod_{k=m}^{n} f_k(x)$ is constant on the interval $(m4^{-n-1}, (m + 1)4^{-n-1}]$ for
$m$ an integer and on adjacent intervals of this type differs by a factor of $3$, $1/3$ or $1$.

3. On $(\frac{1}{2}4^{-n}, 4^{-n}]$, $W(x) = 3^k 2^{-2n-1}$ on a subset of measure
$(\frac{n+1}{k})2^{-3n-2}$.

4. If $0 < x < 1$, $-x/2 \leq h \leq x$ and $0 \leq \theta \leq 1$, then

$$W_b(x)/324 \leq W_b(x + \theta h) \leq 729W_b(x)/4.$$  

Because of periodicity it is sufficient to prove observation 1 for the case
$a = 0$. On an interval $I_i = (i4^{-n}, (i + 1)4^{-n}]$, $\prod_{k=m}^{n-1} f_k(x)$ is constant, and
the integral of $f_n(x)$ is $4^{-n}$. Therefore,

$$\int_{I_i} \prod_{k=m}^{n} f_k(x) dx = \int_{I_i} \prod_{k=m}^{n-1} f_k(x) dx.$$ 

Adding these for $0 \leq i < 4^{n-j}$ shows that

$$\int_0^{4^{-j}} \prod_{k=m}^{n} f_k(x) dx = \int_0^{4^{-j}} \prod_{k=m}^{n-1} f_k(x) dx.$$ 

This process then can be repeated to obtain the result.

Observation 2 follows from the fact that $f_k(x)$ changes by a factor of $3$
at the points $(2j + 1)4^{-k-1} = (2j + 1)4^{-n-k}4^{-n-1}$ and these points cannot coincide for different $k$'s.

For observation 3, first note that $f_k(x) = \frac{1}{2}$ for $x$ in $(\frac{1}{2}4^{-n}, 4^{-n}]$ and
$0 \leq k \leq n - 1$. The functions $f_n, \ldots, f_{2n}$ are independently distributed on
$(\frac{1}{2}4^{-n}, 4^{-n}]$ since each has a full period where the preceding one is constant; this independence also follows immediately from (2.2). Furthermore,
each of these functions equals \( \frac{1}{2} \) on half the interval and \( 3/2 \) on the other half. Therefore, on \((4^{-n}, 4^{-n})\), \( \prod_{k=n}^{2n} f_k(x) \) equals \( 3^{k-2^{-n-1}} \) on a subset of measure \((n+1)2^{-3n-2}\). Observation 3 follows immediately from this.

To prove observation 4, fix \( x \) and \( h \) satisfying the conditions and let \( g(y) = \prod_{k=0}^{m(x,b)} f_k(y) \). By observation 2, \( g(y) \) changes by at most a factor of 3 on an interval of length \( 4^{-n(|h|)} \). Since \( |\theta h| \leq |h| \leq 4^{-n(|h|)} \), \( 3^{-4} g(x) \leq g(x + \theta h) \leq 3^{4} g(x) \). Since \(-x/2 \leq h \leq x\), the product for \( W_b(x + \theta h) \) can have at most two more or two fewer \( f_k \)'s in it than \( g(x + \theta h) \) so

\[
2^{-2} g(x + \theta h) \leq W_b(x + \theta h) \leq 3^{2} 2^{-2} g(x + \theta h).
\]

Combining these inequalities and using the fact that \( g(x) = W_b(x) \) proves observation 4.

3. \( W \) satisfies condition A on \([0, 1]\). It will be shown that there are positive constants, \( C_1 \) and \( C_2 \), such that if \( 0 < h < b \) and \( b + h < 1 \), then

\[
C_1 b W_b(b) \leq \int_{b}^{b+h} W(x) \, dx \leq C_2 h W_b(b),
\]

and

\[
C_1 h W_b(b) \leq \int_{b-h}^{b} W(x) \, dx \leq C_2 h W_b(b).
\]

These show that the integrals of \( W \) over two adjacent subintervals of \([0, 1]\) of the same size are comparable. This obviously implies condition A by considering the two halves of the interval that is to be doubled.

To prove the second inequality in (3.1) write the integral as

\[
\int_{b-h}^{b} W(x) \frac{W(x)}{W_b(x)} \, dx.
\]

By observation 4, (3.3) is bounded by

\[
C W_b(b) \int_{b}^{b+h} \frac{W(x)}{W_b(x)} \, dx;
\]

as noted in §1, \( C \) will denote a positive constant not necessarily the same at each occurrence. By (2.3), (2.4) and (2.5), (3.4) is bounded by

\[
C W_b(b) \int_{b}^{b+4^{-n(b)}} \prod_{k=m(b,h)+1}^{2n(b)} f_k(x) \, dx.
\]
Then observation 1 and the fact that $4^{-n(b)} \leq 4h$ complete the proof of the right inequality in (3.1). For the left inequality in (3.1), proceed in the same manner, and use the first inequality in observation 4 to show that (3.3) is bounded below by (3.4) with a different positive $C$. Next, (3.4) is bounded below by

$$CW_b(b) \int_b^{b+4^{-n(b)-1}} \prod_{k=m(b,b)+1}^{2n(b)} f_k(x) \, dx.$$  

(3.6)

Observation 1 and the fact that $4^{-n(b)} \geq h/4$ complete this part.

If $0 < h \leq b/2$, the inequalities (3.2) follow immediately from (3.1) and property 4. If $b/2 < h < b$, then by the case just observed

$$\int_{b-h}^{b} W(x) \, dx \geq \int_{b/2}^{b} W(x) \, dx \geq C \frac{b}{2} W_{b/2}(b).$$

By the definition of $W_b$, this is bounded below by $ChW_{b/2}(b)$, and the first inequality in (3.2) is proved in this case also. Finally, to prove the second inequality in (3.2) if $b/2 < h < b$, use the fact that

$$\int_{b}^{b-h} W(x) \, dx \leq \int_{0}^{4^{-n(b)}} W(x) \, dx = \sum_{j=n(b)}^{\infty} \int_{4^{-j-1}}^{4^{-j}} W(x) \, dx.$$  

(3.7)

Now on $[4^{-j-1}, 4^{-j}]$, $f_k(x) = \frac{1}{2}$ for $0 \leq k \leq j - 1$ and $f_j(x) \leq 3/2$. Therefore, by observation 1,

$$\int_{4^{-j-1}}^{4^{-j}} W(x) \, dx \leq 3 \cdot 2^{-j} \int_{4^{-j-1}}^{4^{-j}} \prod_{k=j+1}^{2j} f_k(x) \, dx = 9 \cdot 2^{-3j-2},$$

and by (3.7)

$$\int_{b}^{b-h} W(x) \, dx \leq C \cdot 2^{-3n(b)}.$$  

(3.8)

Since $b/2 < h \leq b$, $4^{-n(b)-2} < h \leq 4^{-n(b)}$, and $n(h)$ equals $n(b)$ or $n(b) + 1$. Therefore, $2^{-2n(b)} \leq 4 \cdot 2^{-2n(b)}$, and the definitions of $W_b$ and $f_k$ show that $2^{-n(b)} \leq 2 \prod_{k=0}^{n(b)} f(k) \leq 4W_b(b)$. Writing the right side of (3.8) as $C \cdot 2^{-n(b)}2^{-2n(b)}$ and using these two facts complete the proof of the right inequality in (3.2)

4. $W$ does not satisfy $A_{\infty}$ on $[0, 1]$. This will be shown by producing a sequence of intervals, $I_n$, and a sequence of sets, $E_n$, such that

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To do this let $I_n = [\frac{1}{2}2^{-n}, 2^{-n}]$ and let $E_n$ be the subset of $I_n$ where $W(x) > 3^{2n/3}2^{-2n-1}$. By observation 3,

$$\frac{|E_n|}{|I_n|} = \sum_{k=[2n/3]+1}^{n+1} \binom{n+1}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^n 1 - k,$$

and

$$\int_{E_n} W \int_{I_n} W = \sum_{k=[2n/3]+1}^{n+1} \binom{n+1}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^n 1 - k,$$

where $[2n/3]$ denotes the greatest integer less than or equal to $2n/3$. That the limits of these two ratios are 0 and 1 respectively follows immediately from the fact [2, (16.4.6), p. 200] that if $0 < p < 1$, $q = 1 - p$ and $a < b$, then

$$\lim_{n \to \infty} \sum_{n^p+a(npq)^{1/2}}^{n+b(npq)^{1/2}} \binom{n}{k} p^k q^{n-k} = (2\pi)^{-\frac{1}{2}} \int_a^b e^{-x^2/2} dx.$$

REFERENCES


