ON FUNDAMENTAL TRANSVERSAL MATROIDS¹

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ABSTRACT. Unified proofs of two theorems on fundamental transversal matroids are presented. A necessary condition for a matroid to be a fundamental transversal matroid with respect to a given basis is given.

The purpose of this note is to prove a theorem about fundamental transversal matroids which in turn yields unified proofs of two recent theorems about such matroids. We assume the reader has a modest acquaintance with the basic notions of matroid theory. Thus he or she should know that a matroid (or combinatorial pregeometry) $M$ on a finite set $E$ consists of a nonempty, hereditary collection of subsets of $E$ which are called independent sets, and that the maximal independent sets all have the same cardinality (the rank of $M$) and are called bases. Corresponding to a matroid $M$ there are circuits (minimal nonindependent sets) and a dual matroid $M^*$ whose bases (cobases of $M$) are the complements of the bases of $M$. The circuits of $M^*$ are called cocircuits (or bonds) of $M$. A property important in what follows is that a cocircuit and circuit cannot have exactly one element in common. A matroid is nonseparable if every pair of elements of $E$ lie in a common circuit; otherwise there is a partition $E=I_1 \cup \cdots \cup I_t$ of $E$ and matroids $M_i$ on $E_i$ ($1 \leq i \leq t$) such that $M$ is the direct sum $M_1 \oplus \cdots \oplus M_t$. For more details one can consult [2], [4], or [7].

An interesting kind of matroid is a transversal matroid [3] whose independent sets are the partial transversals of a family of sets. If $(A_i : i \in I)$ is a family of subsets of set $E$, then $T(A_i : i \in I)$ denotes the corresponding transversal matroid.

Suppose $M$ is a matroid on $E$ and $B$ is a basis of $M$. Each $e \in B$ gives rise to a unique cocircuit $C_e^*$ with $e \in C_e^* \subseteq (E \setminus B) \cup \{e\}$ called the fundamental cocircuit of $e$ with respect to the cobasis $E \setminus B$. For $e \in E \setminus B$
set $C^*_e = \{e\}$. Dually each $x \in E \setminus B$ gives rise to a unique circuit $C_x$ with $x \in C_x \subseteq B \cup \{x\}$, called the fundamental circuit of $x$ with respect to the basis $B$. For $x \in B$ set $C_x = \{x\}$. The matroid $M$ is a fundamental transversal matroid [1] if there is a basis $B$ such that $M = T(C^*_e : e \in B)$. Examples of such matroids are the so-called free matroids $\mathcal{F}_r (E)$ whose bases consist of all $r$ element subsets of $E$ (any basis works for $B$ in the definition) and the cycle matroid [4] of the graph of Figure 1.

![Figure 1](image)

A basis (spanning tree) $B$ can be used to show this is a fundamental transversal matroid if and only if $c \in B$. For example, if $B = \{a, c, e\}$, then $\{a, b\}, \{b, c, d\}$, and $\{d, e\}$ give a fundamental presentation. After proving some results we shall look at this example again.

**Theorem 1.** Suppose $M$ is a matroid on $E$ with rank $r$. Let $B$ be a basis of $M$, and $F$ any $r$ element subset of $E$. Then $F$ is an independent set (i.e. basis or transversal) of $T(C^*_e : e \in B)$, denoted $F \in T(C^*_e : e \in B)$, if and only if $B \in T(C^*_e : e \in F)$.

**Proof.** Suppose $B \in M(C_x : x \in F)$, say $B = \{b_x : b_x \in C_x, x \in F\}$. If for some $b_x \in B \setminus F$, $x \notin C^*_{b_x}$, then $|C^*_{b_x} \cap C_x| = |b_x| = 1$. Since this cannot happen, $x \in C^*_{b_x}$ for all $x \in F \setminus B$. If $x \in F \cap B$, then $x \in C_x$, so $F \in T(C^*_e : e \in B)$. The converse is proved in a dual manner.

Applying Theorem 1 to the dual matroid $M^*$ of $M$ we obtain

**Theorem 1*. If $B$ is a basis of the matroid $M$ of rank $r$ on $E$ and $F$ is an $r$ element subset of $E$, then $E \setminus F \in T(C^*_x : x \in E \setminus B)$ if and only if $E \setminus B \in T(C^*_e : e \in E \setminus F)$.

For matroids $M, M'$ of equal rank on the set $E$, recall that $M \subseteq M'$ means that all independent sets (equivalently bases) of $M$ are independent sets of $M'$. In the terminology of [2] this means that the identity map on $E$ is a geometric (weak) map.

**Corollary 1.** $M \subseteq T(C^*_e : e \in B)$.

**Proof.** Let $F$ be any basis of $M$. Suppose $B \notin T(C^*_x : x \in F)$. Then by
Hall's theorem (see [6]), there exists $G \subseteq F$ with $\bigcup_{x \in G} C_x \cap B < |G|$. But then $\bigcup_{x \in G} C_x \cap B$ and $G$ are both independent subsets of $\bigcup_{x \in G} C_x$ with the former a maximal independent subset and the latter of larger cardinality. This contradicts a basic tenet of matroid theory. The proof is now completed by invoking Theorem 1.

**Corollary 2.** Let $F$ be an $r$ element subset of $E$ which is not a basis of $M$. Then there is a basis $B$ of $M$ (any extension to a basis of a maximal independent subset $A$ of $F$ will do) such that $F \notin T(C_e^* : e \in B)$.

**Proof.** $C_x \subseteq A$ for all $x \in F$. Hence $B \notin T(C_x : x \in F)$, so by Theorem 1, $F \notin T(C_e^* : e \in B)$.

The matroid $M$ is the basis intersection of matroids $M_1, \ldots, M_r$ provided the bases of $M$ are precisely the common bases of $M_1, \ldots, M_r$.

**Corollary 3** (Bondy and Welsh [1]). Every matroid of rank $r$ is the basis intersection of fundamental transversal matroids of rank $r$.

**Proof.** Corollaries 1 and 2.

**Theorem 2.** Let $F$ be an $r$ element subset of $E$. Then $F \in T(C_e^* : e \in B)$ if and only if $E \setminus B \in T(C_e^* : e \in E \setminus F)$.

**Proof.** Since for $x \in F \cap B$, the only member of the family $(C_e^* : e \in B)$ which contains $x$ is $C_x^*$, we conclude that

(i) $F \in T(C_e^* : e \in B)$ if and only if $F \setminus B \in T(C_e^* : e \in B \setminus F)$. Now $E \setminus B = (F \setminus B) \cup X$ and $E \setminus F = (B \setminus F) \cup X$ where $X = (E \setminus F) \cap (E \setminus B)$. Since for $e \in X$, $C_e^* = \{e\}$ we conclude that

(ii) $E \setminus B \in M(C_e^* : e \in E \setminus F)$ if and only if $F \setminus B \in T(C_e^* : e \in B \setminus F)$. Combining (i) and (ii) we obtain the conclusion of the theorem.

**Corollary 4.** Let $F$ be an $r$ element subset of $E$. Then $B \in T(C_x : x \in F)$ if and only if $E \setminus B \in T(C_e^* : e \in E \setminus F)$.

**Proof.** Theorems 1 and 2.

**Corollary 5** (Las Vergnas [5]). The dual of a fundamental transversal matroid is a fundamental transversal matroid.

**Proof.** Let $M = T(C_e^* : e \in B)$. Let $F$ be any $r$ element subset of $E$. Then

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Thus $M^* = T(C_x^*: x \in E \setminus B)$.

**Example.** Let $M$ be the cycle matroid of the graph which is obtained from a triangle by doubling each edge. It is well known and easy to check that $M$ is not a transversal matroid. In light of Corollary 5 this means that $M^*$ is not a fundamental transversal matroid (although it is a transversal matroid), since $M = (M^*)^*$.

**Theorem 3.** Let $M$ be a matroid on $E$ and let $B$ be a basis. Suppose for some $y \in E \setminus B$ there is a circuit $D$ with $y \in D$ and $|D| < |C_y|$. Then $M \neq T(C_e^*: e \in B)$.

**Proof.** Consider the family $\{C_x \cap B: x \in D\}$. Since every proper subset $A$ of $D$ is independent, for all $A \subseteq D$, $\bigcup_{x \in A} C_x \cap B \geq |A|$. But also $|\bigcup_{x \in D} C_x \cap B| \geq |C_y \cap B| = |C_y| - 1 \geq |D|$.

Thus by Hall's theorem there is a subset $T$ of $B$ which is a transversal of $(C_x \cap B: x \in D)$. Let $F$ be the $r$ element set $D \cup (B \setminus T)$. Then since $C_x = \{x\}$ for $x \in B \setminus T$, $B \in T(C_x^*: x \in F)$. So by Theorem 1, $F \in T(C_e^*: e \in B)$.

Since $D \subseteq F$ and $D$ is a circuit of $M$, $M \neq T(C_e^*: e \in B)$.

If $M$ is the cycle matroid of the graph of Figure 1, then while $M$ is a fundamental transversal matroid $M \neq T(C_x^*: x \in B)$ for the basis $B = \{a, b, e\}$. We can say this since $C_d = \{a, b, d, e\}$ while $D = \{d, c, e\}$ is a smaller circuit containing $d$.

As a final result we characterize those matroids which are fundamental transversal matroids with respect to every basis.

**Theorem 4.** Let $M$ be a nonseparable matroid of rank $r$ on $E$ such that $M = T(C_e^*: e \in B)$ for all bases $B$. Then $M$ is the matroid $P_r(E)$.

**Proof.** From Theorem 3 we conclude that for each $y \in E$ all circuits containing $x$ have the same cardinality. This is so since if $C$, $D$ were circuits containing $y$ with $|D| < |C|$, we could choose a basis $B$ with $C \setminus y \subseteq B$. For this basis $B$, $C_y = C$. 

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ON FUNDAMENTAL TRANSVERSAL MATROIDS

155

Since $\mathcal{M}$ is nonseparable each pair of elements of $E$ lies in a circuit. Thus all circuits of $\mathcal{M}$ have the same cardinality, say $k$. Let $B$ be any basis of $\mathcal{M}$. Let $x, y \in E \setminus B$ and choose a circuit $D$ containing $x$ and $y$. If $C_x \setminus \{x\} \neq C_y \setminus \{y\}$, then

$$\left| \bigcup_{x \in D} C_x \cap B \right| \geq |(C_x \setminus \{x\}) \cup (C_y \setminus \{y\})| \geq |C_x| = |D|$$

and as in the proof of Theorem 3 we conclude that $\mathcal{M} \neq T(C^*_e; e \in B)$. Thus $C_x \setminus \{x\} = C_y \setminus \{y\}$ for all $x, y \in E \setminus B$. But since $\mathcal{M}$ is nonseparable, this readily implies that $C \setminus \{x\} = B$ for all $x \in E \setminus B$. In particular $k = r + 1$.

Now let $F$ be any $r$ element subset of $E$. Since all circuits of $\mathcal{M}$ have cardinality $r + 1$, $A$ is a basis of $\mathcal{M}$. Hence $\mathcal{M} = \mathcal{P}_r(E)$.

**Corollary 6.** If $\mathcal{M}$ is a matroid of rank $r$ on $E$ such that $\mathcal{M} = \mathcal{M}(C^*_e; e \in B)$ for all bases $B$, then there is a partition $E_1, \ldots, E_t$ of $E$ with $\mathcal{M} = \mathcal{P}_{r_1}(E_1) \oplus \cdots \oplus \mathcal{P}_{r_t}(E_t)$ where $r = r_1 + \cdots + r_t$.

**Example.** Let $\mathcal{M}$ be the cycle matroid of the graph in Figure 2.

```
\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (b) at (1,0) {$b$};
  \node (c) at (0,-1) {$c$};
  \node (d) at (1,-1) {$d$};
  \node (s) at (0,1) {$s$};
  \node (t) at (1,1) {$t$};
  \node (u) at (0,-2) {$u$};
  \draw (a) -- (b);
  \draw (a) -- (c);
  \draw (a) -- (d);
  \draw (b) -- (c);
  \draw (b) -- (d);
  \draw (c) -- (d);
\end{tikzpicture}
\caption{Figure 2}
\end{figure}
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With respect to the basis $B = \{a, b, c, d\}$, for each $x \notin B$, $C_x$ has smaller cardinality (namely 3) than any other circuit containing $x$. Thus the possibility that $\mathcal{M} = T(C^*_x; x \in B)$ is not ruled out by Theorem 3. But if $D = \{s, t, u, v\}$, then $D$ is a circuit of $\mathcal{M}$ and $D \in T(C^*_x; x \in B)$. (This is most readily checked by using Theorem 1.) Thus $\mathcal{M} \neq T(C^*_x; x \in B)$.

**REFERENCES**


