PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 2, August 1974

ANOTHER PROOF OF SZEGÖ'S THEOREM FOR A SINGULAR MEASURE

FINBARR HOLLAND

ABSTRACT. It is shown that the set $\{e^{int}: n \geq 1\}$ spans $\mathfrak{L}^2(\sigma)$ if σ is a singular measure on the unit circle. The proof makes no appeal either to the F. and M. Riesz theorem on measures or to Hilbert space methods.

Theorem. *If σ is a singular probability measure on the unit circle, then the functions e^{int}, $n \geq 1$, span $\mathfrak{L}^2(\sigma)$.*

Proof. As usual, it is enough to show that the closed linear span of the set $\{e^{int}: n \geq 1\}$ in $\mathfrak{L}^2(\sigma)$ contains the constant function 1. To establish this, a sequence of analytic polynomials $\{P_n\}$, with $P_n(0) = 0$ and $\deg P_n \leq n$, is constructed so that $\lim \int |1 - P_n|^2 d\sigma = 0$.

To begin with, set

$$A_1 = \int e^{-it}\, d\sigma(t) \quad \text{and} \quad A_k = \int e^{-ikt} \left[1 - \sum_{j=1}^{k-1} A_j e^{ijt} \right] d\sigma(t)$$

for $k \geq 2$. Then

$$2A_1 = C_1 \quad \text{and} \quad 2A_k = C_k - \sum_{j=1}^{k-1} A_j C_{k-j} \qquad (k \geq 2), \tag{1}$$

where $C_k = 2\int e^{-ikt}\, d\sigma(t)$.

Now consider the sequence of polynomials

$$P_n(z) = \sum_{k=1}^{n} A_k z^k \qquad (n = 1, 2, \cdots).$$

An induction argument shows that

$$\int |1 - P_n|^2\, d\sigma = 1 - \sum_{k=1}^{n} |A_k|^2 \qquad (n = 1, 2, \cdots), \tag{2}$$

Received by the editors September 11, 1973.

AMS (MOS) subject classifications (1970). Primary 30A18; Secondary 30A76.

Key words and phrases. Singular probability measure, closed linear span, inner function, Hardy space H^2.

whence it follows that $\Omega(z) = \sum_{k=1}^{\infty} A_k z^k$ belongs to the Hardy space H^2. In view of (2), the Theorem follows once we have shown that Ω is an inner function. With this in mind, we introduce

$$F(z) = 1 + \sum_{k=1}^{\infty} C_k z^k = \int \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\sigma(t) \qquad (|z| < 1).$$

Clearly, F is regular on the open unit disc and its real part is positive there. Also, it is easy to conclude from (1) that

$$F(z) = \frac{1 + \Omega(z)}{1 - \Omega(z)} \quad \text{and} \quad \operatorname{Re} F(z) = \frac{1 - |\Omega(z)|^2}{|1 - \Omega(z)|^2} \qquad (|z| < 1).$$

Since σ is singular, Re F vanishes on the circle outside a set of Lebesgue measure zero [1, p. 34]. Equivalently, $|\Omega| = 1$ almost everywhere on the circle. Thus, Ω is inner and the proof is finished.

REFERENCE

1. K. Hoffmann, *Banach spaces of analytic functions*, Prentice-Hall Series in Modern Analysis, Prentice-Hall, Englewood Cliffs, N. J., 1962. MR 24 #A2844.

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE, CORK, IRELAND (Current address)

DEPARTMENT OF MATHEMATICS, CHELSEA COLLEGE, LONDON, ENGLAND