

ENUMERATION OF POSETS GENERATED BY DISJOINT UNIONS AND ORDINAL SUMS¹

RICHARD P. STANLEY

ABSTRACT. Let f_n be the number of n -element posets which can be built up from a given collection of finite posets using the operations of disjoint union and ordinal sum. A curious functional equation is obtained for the generating function $\sum f_n x^n$. Using a result of Bender, an asymptotic estimate can sometimes be given for f_n . The analogous problem for labeled posets is also considered.

Let P and Q be partially ordered sets (or posets). Regard P and Q as being relations on two disjoint sets T and T' , respectively. The *disjoint union* $P + Q$ is defined to be the partial ordering on $T \cup T'$ satisfying: (1) If $x \in T$, $y \in T$, and $x \leq y$ in P , then $x \leq y$ in $P + Q$; (2) if $x \in T'$, $y \in T'$, and $x \leq y$ in Q then $x \leq y$ in $P + Q$. The *ordinal sum* $P \oplus Q$ is defined to be the partial ordering on $T \cup T'$ satisfying (1), (2), and the additional condition (3) if $x \in T$ and $y \in T'$, then $x \leq y$ in $P \oplus Q$. Hence $+$ is commutative but \oplus is not.

The question we consider is the following. Let S be a set of nonvoid isomorphically distinct finite posets, such that no element P of S is a disjoint union or an ordinal sum of two nonvoid posets. (We say that P is $(+, \oplus)$ -irreducible.) How many isomorphically distinct posets of cardinality n can be built up from the elements of S by the operations of disjoint union and ordinal sum? Call a poset that can be obtained in this way an S -poset. Hence if P and Q are S -posets, then so are $P + Q$ and $P \oplus Q$. Moreover, the $(+, \oplus)$ -irreducible S -posets are simply the members of S . For instance, if S consists of a single one-element poset, then there are two 2-element S -posets and five 3-element S -posets, viz., 21 , 2 , 31 , $2 + 1$, $(21) \oplus 1$, $1 \oplus (21)$, and 3 . Here n denotes an n -element chain and nP a disjoint union $P + \dots + P$ of n copies of P .

Received by the editors April 23, 1973.

AMS (MOS) subject classifications (1970). Primary 05A15; Secondary 06A10.

Key words and phrases. Poset, partially ordered set, disjoint union, ordinal sum, generating function, Pólya's enumeration theorem, functional equation.

¹The research was supported by a Miller Research Fellowship at the University of California at Berkeley.

Let f_n denote the number of S -posets of cardinality n . We set $f_0 = 1$. Define the generating function $F(x) = \sum_{n=0}^{\infty} f_n x^n$. We shall determine a functional equation for $F(x)$. The technique used is analogous to that appearing in [3, Chapter 6, §10] for the enumeration of series-parallel networks. However, instead of a duality principle allowing us to obtain an explicit functional equation for $F(x)$, we instead are helped by the triviality of one of the two groups arising from the enumeration.

If P is an S -poset and P can be written $P_1 + P_2$ where neither P_i is void, then we say P is *essentially +*. Similarly if $P = P_1 \oplus P_2$ where neither P_i is void, we say P is *essentially \oplus* . Every S -poset not a member of S is either essentially + or essentially \oplus , but not both. By convention we agree that every member of S is both essentially + and essentially \oplus .

Let a_n be the number of n -element members of S , so $a_0 = 0$ since we are assuming the members of S to be nonvoid. Let u_n be the number of n -element essentially + S -posets and v_n the number of n -element essentially \oplus S -posets. We define $u_0 = v_0 = 0$. Hence $f_n = u_n + v_n - a_n$ if $n \geq 1$. Define the generating functions

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad U(x) = \sum_{n=0}^{\infty} u_n x^n, \quad V(x) = \sum_{n=0}^{\infty} v_n x^n.$$

It follows that

$$(1) \quad F(x) = U(x) + V(x) - A(x) + 1.$$

Now every essentially + S -poset P not belonging to S can be written *uniquely* as a disjoint union $P_1 + P_2 + \dots + P_m$ of $m \geq 2$ essentially \oplus S -posets P_i , where the order of the P_i 's is immaterial. Let u_{mn} be the number of essentially + S -posets of cardinality n which are the disjoint union of m essentially \oplus S -posets. Define for $m \geq 2$ the generating functions $U_m(x) = \sum_{n=1}^{\infty} u_{mn} x^n$.

Since the order of addition in a disjoint union is immaterial, it follows immediately from Pólya's theorem, as expounded in [3, Chapter 6], that

$$(2) \quad U_m(x) = Z(\mathfrak{S}_m | V(x), V(x^2), V(x^3), \dots), \quad m \geq 2,$$

where $Z(\mathfrak{S}_m | z_1, z_2, z_3, \dots)$ is the cycle index polynomial of the symmetric group \mathfrak{S}_m of degree m . Now $A(x) + \sum_{m=2}^{\infty} U_m(x) = U(x)$. Hence from (2) we obtain

$$(3) \quad U(x) = A(x) + \sum_{m=0}^{\infty} Z(\mathfrak{S}_m | V(x), V(x^2), V(x^3), \dots) - V(x) - 1,$$

License or copyright restrictions may apply to redistribution; see <https://www.ams.org/journal-terms-of-use>
 since $Z(\mathfrak{S}_0 | V(x), V(x^2), \dots) = 1$ and $Z(\mathfrak{S}_1 | V(x), V(x^2), \dots) = V(x)$.

Now it is well known that

$$\sum_{m=0}^{\infty} Z(\mathbb{G}_m | z_1, z_2, z_3, \dots) t^m = \exp(z_1 t + z_2 t^2/2 + z_3 t^3/3 + \dots)$$

(cf. [3, p. 133]). Thus from (3) we get

$$(4) \quad U(x) = \exp \left[\sum_{k=1}^{\infty} \frac{V(x^k)}{k} \right] - V(x) + A(x) - 1.$$

Similarly every essentially \oplus S -poset P not belonging to S can be written *uniquely* as an ordinal sum $P_1 \oplus P_2 \oplus \dots \oplus P_m$ of $m \geq 2$ essentially $+S$ -posets P_i , where now the order of the P_i 's is fixed. Define v_{mn} and $V_m(x)$ for $m \geq 2$ in analogy to u_{mn} and $U_m(x)$. Since now the order of summing cannot be altered, we get from Pólya's theorem that

$$(5) \quad V_m(x) = Z(G_m | U(x), U(x^2), U(x^3), \dots), \quad m \geq 2,$$

where $Z(G_m | z_1, z_2, z_3, \dots)$ is the cycle index polynomial of the trivial group G_m of order one acting on an m -set. Since $Z(G_m | z_1, z_2, \dots) = z_1^m$, we get

$$(6) \quad V_m(x) = U(x)^m, \quad m \geq 2.$$

Of course (6) can be easily obtained directly, but we wanted to make clear the similarity of (2) and (5).

Now

$$V(x) = A(x) + \sum_{m=2}^{\infty} V_m(x) = \sum_{m=0}^{\infty} U(x)^m - U(x) + A(x) - 1,$$

so

$$(7) \quad V(x) = 1/(1 - U(x)) - U(x) + A(x) - 1.$$

Eliminating $U(x)$ from (1) and (7) yields

$$(8) \quad V(x) = F(x) + 1/F(x) - 2 + A(x).$$

Thus by (4) and (8),

$$\begin{aligned} F(x) &= U(x) + V(x) - A(x) + 1 \\ &= \exp \left[\sum_{k=1}^{\infty} \frac{1}{k} \left(F(x^k) + \frac{1}{F(x^k)} - 2 + A(x^k) \right) \right]. \end{aligned}$$

License or copyright restrictions may apply to redistribution; see <https://www.ams.org/journal-terms-of-use>

We have obtained

Theorem 1. $F(x)$ satisfies the functional equation

$$(9) \quad F(x) = \exp \left[\sum_{k=1}^{\infty} \frac{1}{k} \left(F(x^k) + \frac{1}{F(x^k)} - 2 + A(x^k) \right) \right].$$

Since $\exp \left(\sum_{k=1}^{\infty} ax^k/k \right) = (1-x)^{-a}$, (9) may be rewritten in the rather curious form

$$F(x) = \prod_{i=1}^{\infty} (1-x^i)^{-(f_i + g_i + a_i)},$$

where $1/F(x) = 1 + \sum_{i=1}^{\infty} g_i x^i$.

For instance, if S consists of a single one-element poset, then $A(x) = x$ and

$$F(x) = 1 + x + 2x^2 + 5x^3 + 15x^4 + 48x^5 + 167x^6 + \dots$$

Another case of interest is when S consists of all isomorphically distinct finite $(+, \oplus)$ -irreducible posets. Then f_n is the number of isomorphically distinct posets of cardinality n . It can be shown (cf. [4] for the numbers f_n) that here

$$A(x) = x + x^4 + 12x^5 + 104x^6 + 956x^7 + \dots,$$

$$F(x) = 1 + x + 2x^2 + 5x^3 + 16x^4 + 63x^5 + 318x^6 + 2045x^7 + \dots$$

A theorem of Bender [1, Theorem 5] allows the determination of an asymptotic formula for f_n when $A(x)$ is a polynomial. We shall spare the reader the details of the calculations and merely state that when $A(x) = x$, we get $f_n \sim Cn^{-3/2}\alpha^{-n}$, where α is the unique positive root of $F(\alpha) - (1 + \sqrt{5})/2$, and C is a constant given by

$$C = \left(\frac{1}{\pi(3\sqrt{5} - 5)} \left[\frac{\alpha}{1 - \alpha} + \sum_{k=2}^{\infty} \alpha^k F'(\alpha^k) \left(1 - \frac{1}{F(\alpha^k)^2} \right) \right] \right)^{1/2}.$$

Note that not surprisingly f_n is very much smaller than the total number p_n of posets of cardinality n , which by [2] is given by $p_n = 2^{n^2/4 + o(n^2)}$.

One can also ask analogous questions for labeled posets. A labeling of a finite poset P is an injection $\phi: P \rightarrow \mathbf{Z}$. Two labelings, ϕ and ψ , of P are equivalent if there is an automorphism $\rho: P \rightarrow P$ and an order-preserving injection $\sigma: \mathbf{Z} \rightarrow \mathbf{Z}$ such that $\phi = \sigma\psi\rho$. Let S be a set of nonvoid inequivalent $(+, \oplus)$ -irreducible labeled finite posets. Let h_n be the number of n -element inequivalent labeled posets (P, ϕ) that can be built up from the members of S by the operations of $+$ and \oplus , such that the restriction of ϕ to any $(+, \oplus)$ -irreducible component of P is equivalent to the labeling of a member of S . (We set $h_0 = 1$ by convention.) Let b_n be the number of n -element members of S (so $b_0 = 0$). Define the exponential generating functions

$$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}, \quad H(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!}.$$

Then by an argument analogous to that used to prove Theorem 1, which we omit, one obtains

Theorem 2. *H(x) satisfies the functional equation*

$$H(x) = \exp[H(x) + 1/H(x) - 2 + B(x)].$$

REFERENCES

1. E. A. Bender, *Asymptotic methods in enumeration*, SIAM Rev. (to appear).
2. D. Kleitman and B. Rothschild, *The number of finite topologies*, Proc. Amer. Math. Soc. 25(1970), 276–282. MR 40 #7157.
3. J. Riordan, *An introduction to combinatorial analysis*, Wiley, New York; Chapman & Hall, London, 1958. MR 20 #3077.
4. J. Wright, *Cycle indices of certain classes of quasiorder types or topologies*, Dissertation, University of Rochester, Rochester, N. Y., 1972.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720

Current address: Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139