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SEMIHEREDITARY POLYNOMIAL RINGS

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ABSTRACT. It is shown that if the ring of polynomials over a commutative ring R is semihereditary then R is von Neumann regular. This is the converse of a theorem of P. J. McCarthy.

P. J. McCarthy [2] has recently proved that for the ring of polynomials $R[x]$ over a commutative ring to be semihereditary, it is sufficient that R be von Neumann regular. The purpose of this note is to show that this condition actually characterizes von Neumann regular rings.

The lattice of ideals of a commutative von Neumann regular ring is distributive, but not all such rings are von Neumann regular. If the lattice of ideals of $R[x]$ is distributive then the lattice of ideals of R is, since R is a homomorphic image of $R[x]$. In the process of proving the converse of McCarthy's theorem, we show that this latter condition characterizes von Neumann regular rings. Summarizing:

Theorem. *The following are equivalent for a commutative ring R:*
1. *R is von Neumann regular.*
2. *$R[x]$ is semihereditary.*
3. *$R[x]$ has a distributive lattice of ideals.*

Proof. 1 implies 2 is McCarthy's theorem. The fact that a commutative semihereditary ring has a distributive lattice of ideals may be found in [1], which yields 2 implies 3.

To show 3 implies 1, we use the fact that a ring R has a distributive lattice of ideals if and only if, for $r, s \in R$, $(r:s)+(s:r)=R$ where $(s:r)=\{x \in R \mid sx \in rR\}$ [1]. The above statement is easily seen to be equivalent to the existence of u, v, and $w \in R$ with: $r(1-u)=sv$ and $su=rw$. Now, let $a \in R$. We must show $a^2R=aR$. The fact that $R[x]$ has a distributive lattice of ideals yields $u(x)$, $v(x)$ and $w(x)$ with:

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(1) $xu(x) = av(x)$,

(2) $a[1 - u(x)] = xw(x)$.

Multiplying both sides of (2) by x, we obtain $ax - axu(x) = x^2w(x)$, and using (1) to substitute $av(x)$ for $xu(x)$ in this we have:

(3) $ax - a^2v(x) = x^2w(x)$.

But, if v_1 is the x coefficient of v, then the x coefficient of the left side of (3) is $a - a^2v_1$, while the x coefficient of the right side is zero, so $a = a^2v_1$ and we are done.

Remark. The above proof actually shows that if $I = aR[x] + xR[x]$ is projective, then aR is generated by an idempotent. Since Lemma 1 of [1] asserts that if R is commutative and $xR + yR$ is projective, then $(x:y) + (y:x) = R$. The converse is also true, for if $aR = eR$, then $((1 - e)/x)I \subset R$, and $1 = ((1 - e)/x)x + e$, so that I is invertible.

BIBLIOGRAPHY

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