SEMIHEREDITARY POLYNOMIAL RINGS

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ABSTRACT. It is shown that if the ring of polynomials over a commutative ring \( R \) is semihereditary then \( R \) is von Neumann regular. This is the converse of a theorem of P. J. McCarthy.

P. J. McCarthy [2] has recently proved that for the ring of polynomials \( R[x] \) over a commutative ring to be semihereditary, it is sufficient that \( R \) be von Neumann regular. The purpose of this note is to show that this condition actually characterizes von Neumann regular rings.

The lattice of ideals of a commutative von Neumann regular ring is distributive, but not all such rings are von Neumann regular. If the lattice of ideals of \( R[x] \) is distributive then the lattice of ideals of \( R \) is, since \( R \) is a homomorphic image of \( R[x] \). In the process of proving the converse of McCarthy's theorem, we show that this latter condition characterizes von Neumann regular rings. Summarizing:

**Theorem.** The following are equivalent for a commutative ring \( R 

1. \( R \) is von Neumann regular.
2. \( R[x] \) is semihereditary.
3. \( R[x] \) has a distributive lattice of ideals.

**Proof.** 1 implies 2 is McCarthy's theorem. The fact that a commutative semihereditary ring has a distributive lattice of ideals may be found in [1], which yields 2 implies 3.

To show 3 implies 1, we use the fact that a ring \( R \) has a distributive lattice of ideals if and only if, for \( r, s \in R \), \( (r : s) + (s : r) = R \) where \( (s : r) = \{ x \in R \mid sx \in rR \} \) [1]. The above statement is easily seen to be equivalent to the existence of \( u, v, \) and \( w \in R \) with: \( r(1 - u) = sv \) and \( su = rw \). Now, let \( a \in R \). We must show \( a^2R = aR \). The fact that \( R[x] \) has a distributive lattice of ideals yields \( u(x), v(x) \) and \( w(x) \) with:

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(1) \( xu(x) = a \nu(x) \),
(2) \( a[1 - u(x)] = xw(x) \).

Multiplying both sides of (2) by \( x \), we obtain \( ax - axu(x) = x^2w(x) \), and using (1) to substitute \( a \nu(x) \) for \( xu(x) \) in this we have:

(3) \( ax - a^2 \nu(x) = x^2w(x) \).

But, if \( \nu_1 \) is the \( x \) coefficient of \( \nu \), then the \( x \) coefficient of the left side of (3) is \( a - a^2 \nu_1 \), while the \( x \) coefficient of the right side is zero, so \( a = a^2 \nu_1 \) and we are done.

Remark. The above proof actually shows that if \( I = aR[x] + xR[x] \) is projective, then \( aR \) is generated by an idempotent. Since Lemma 1 of [1] asserts that if \( R \) is commutative and \( xR + yR \) is projective, then \( (x:y) + (y:x) = R \). The converse is also true, for if \( aR = eR \), then \( ((1 - e)/x) \subset R \), and \( 1 = ((1 - e)/x)x + e \), so that \( I \) is invertible.

BIBLIOGRAPHY