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SOME INEQUALITIES FOR THE MULTIPLICATOR OF A FINITE GROUP. II

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ABSTRACT. In a previous article of the same title we gave a series of upper and lower bounds for the multiplicator of a finite group. §2 of this article gives an improvement of one of these results. In §3, we give a result that connects the multiplicator of a finite group with that of a normal subgroup whose factor-group is cyclic.

1. **Introduction.** In [3] we gave a result that connected the multiplicator, $M(G)$, of a finite nilpotent group G with the terms of the lower central series for G (Theorem 4.4). This result has already been generalized by Vermani [7] although we were unaware of this fact at the time of publication. In §2 we use methods similar to those of [3] to give a generalization of Vermani's result.

In §3 we give a result that connects the multiplicator of a finite group with that of a normal subgroup.

Notation. (i) *If* X *is a finite group then* $e(X)$ *denotes the exponent of* X *and* $d(X)$ *denotes the minimal number of generators of* X.

(ii) *The lower central series of a group* G *will be denoted by* $G = G_1 \geq G_2 = G' \geq G_3 \geq \cdots$, *and the upper central series will be denoted by* $1 = Z_0(G) \leq Z_1(G) = Z(G) \leq Z_2(G) \leq \cdots$.

All other notation, where not explained, will be standard, or may be found in [3].

We will need the following result for the work of the next section:

1.1 (**Schur** [6]). *Let* G *be a finite group and* $G = F/R$ *a presentation for* G *with* F *free. Then* $M(G) \cong (F' \cap R)/[F, R]$.

2. **Central series.** The main result of this section is the following generalization of [3, Theorem 4.4]:

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Theorem 2.1. *Let G be a finite nilpotent group of class $c \geq 2$ and let $G = F/R$ be a presentation for G as a factor-group of the free group F. Set $Q = G/G_c$. Then*

(i) $|G_c|\,|M(G)| = |M(Q)|\,|[F, F_cR]/[F, R]|$,

(ii) $d(M(G)) \leq d(M(Q)) + d([F, F_cR]/[F, R])$,

(iii) $e(M(G)) \leq e(M(Q))e([F, F_cR]/[F, R])$.

Proof. We have $G_c = F_cR/R \cong F_c/(F_c \cap R)$ and $Q \cong F/F_cR$ so that

$$M(Q) \cong (F' \cap F_cR)/[F, F_cR] = (F' \cap R)F_c/[F, F_cR].$$

Hence

$$|(F' \cap R)F_c/[F, R]| = |M(Q)|\,|[F, F_cR]/[F, R]|,$$

$$\begin{aligned} d(M(G)) &\leq r((F' \cap R)F_c/[F, R]) \\ &\leq r(M(Q)) + r([F, F_cR]/[F, R]) \\ &= d(M(Q)) + d([F, F_cR]/[F, R]), \end{aligned}$$

and

$$e(M(G)) \leq e(M(Q))e([F, F_cR]/[F, R]).$$

But

$$((F' \cap R)F_c/[F, R])/((F' \cap R)/[F, R]) \cong F_c/(F_c \cap R) \cong G_c.$$

The required result now follows.

As mentioned in the introduction, Theorem 2.1 is a generalization of a result of [7]. To see this we need a simple lemma.

For the remainder of this section, unless otherwise stated, we let $G = F/R$ be a presentation of the finite group G, nilpotent of class $c \geq 2$. For convenience we set $Z_j = Z_j(G)$ for $0 \leq j \leq c$ so that $Y_0 = R$ and $Y_c = F$. It follows at once that for $1 \leq j \leq c$, $[Y_j, F] \leq Y_{j-1}$, and $[F_j, Y_j] \leq R$.

Lemma 2.2. *For y in Y_{c-1} and x in F_c we have that $[y, x] \equiv 1$ $(\mathrm{mod}[F, R])$.*

Proof. Consider the series

$$F \geq \cdots \geq Y_k \geq Y_{k-1} \geq \cdots \geq Y_1 \geq R \geq [F, R].$$

Then by the preceding remark and [1, III, 2.8], $[Y_k, F_{k+1}] \leq [F, R]$ for all k. In particular, $[Y_{c-1}, F_c] \leq [F, R]$, so that the lemma is established.

Lemma 2.3. *In the above notation,* $[F, F_cR]/[F, R]$ *is an epimorphic image of* $(G/Z_{c-1}) \otimes G_c$.

Proof. We proceed as in Lemma 2.1 of [2]. Define θ on $(F/Y_{c-1}) \times (F_cR/R)$ by the rule $(fY_{c-1}, xR)\theta = [f, x][F, R]$ for f in F and x in F_c.

Suppose $f_1 = fy$ and $x_1 = xr$ for y in Y_{c-1} and r in R. Then the usual commutator calculations and Lemma 2.2 show that $[f_1, x_1] \equiv [f, x]$ (mod $[F, R]$) and θ is well defined.

To complete the proof we now need only remark that for f, f_1, f_2 in F and x, x_1, x_2 in F_c,

$$[f_1 f_2, x] \equiv [f_1, x][f_2, x] \pmod{[F, R]}$$

and

$$[f, x_1 x_2] \equiv [f, x_1][f, x_2] \pmod{[F, R]}.$$

Proposition 2.4. *Let* G *be a finite nilpotent group of class* $c \geq 2$, *let* $Z_j = Z_j(G)$ *for all* j *and let* $Q = G/G_c$. *Then*

(i) $|G_c|\ |M(G)| \leq |M(Q)|\ |(G/Z_{c-1}) \otimes G_c|$,

(ii) $d(M(G)) \leq d(M(Q)) + d((G/Z_{c-1}) \otimes G_c)$,

(iii) $e(M(G)) \leq e(M(Q))e((G/Z_{c-1}) \otimes G_c)$.

Proof. This follows at once from Theorem 2.1 and Lemma 2.3.

Part (i) of Proposition 2.4 is due to Vermani [7].

As an application of Proposition 2.4 (iii) we give an improvement of [3, Corollary 4.6] for p-groups of class at least 2.

It has been conjectured that the exponent of the multiplicator of a finite p-group is a divisor of the exponent of the group itself. The next lemma shows the conjecture to be true for p-groups of class 2. The results can be extended to some p-groups of class 3 and some of the possible extensions are discussed in a remark to follow. However, a remarkable example of a group of exponent 4 whose multiplicator has exponent 8 shows that the conjecture is, in general, false. This counterexample has been constructed using computer techniques by I. D. Macdonald, J. W. Wamsley and others.

Lemma 2.5. *Let* G *be a finite* p*-group of class* 2 *and let* H *be a representing group for* G. *Then* $e(H')$ *is a divisor of* $e(G)$.

Proof. For the definition of representing groups see [1] or [5]. Let G have exponent p^e and let $G = \langle x_1, \cdots, x_t\rangle$, where $t = d(G)$. Then, if for $1 \leq j \leq t$, y_j is a pre-image of x_j in H, $H = \langle y_1, \cdots, y_t\rangle$.

Now H contains a subgroup L in $Z(H) \cap H'$ such that $H/L \cong G$ and

$L \cong M(G)$. Clearly we have

$$G' = \langle [x_i, x_j] \backslash 1 \le i < j \le t \rangle,$$

$$H' = \langle [y_i, y_j], H_3 \backslash 1 \le i < j \le t \rangle,$$

and

$$H_3 = \langle [y_i, y_j, y_k] \backslash i \le k \le t, 1 \le i < j \le t \rangle \le Z(H).$$

Further, H is metabelian. We consider two cases.

Case 1. *p odd.* For all j, $y_j^{p^e} \in L \le Z(H)$. Hence for all i and j we have

$$1 = [y_i, y_j^{p^e}] = [y_i, y_j]^{p^e} [y_i, y_j, y_j]^{p^e(p^e-1)/2}.$$

Further, for $i \le k \le t$, $1 \le i < j \le t$, $[y_i, y_j]^{p^e} \in L$ so that

$$1 = [[y_i, y_j]^{p^e}, y_k] = [y_i, y_j, y_k]^{p^e}.$$

The desired result now follows.

Case 2. $p = 2$. Since G is nonabelian we must have $e \ge 2$. For all i and j we have

$$1 = (x_i x_j)^{2^e} = x_i^{2^e} x_j^{2^e} [x_j, x_i]^{2^{e-1}(2^e-1)}.$$

Hence $[x_i, x_j]^{2^{e-1}} = 1$ and the result follows as for case 1.

Corollary 2.6. *Let G be a finite p-group of class 2 and exponent p^e. Then $e(M(G)) \le p^e$.*

Corollary 2.7. *Let G be a p-group of class $c \ge 2$ and suppose $e(G) = p^e$. Then $e(M(G)) \le p^{e(c-1)}$.*

Proof. This follows by induction using Corollary 2.6 and Proposition 2.4 (iii).

Remark 2.8. The result of Lemma 2.5 is probably well known but we know of no reference to it. The proof can obviously be extended to some groups of class 3; all that is needed is the following commutator identity which holds in all groups of class no more than 4:

$$[a, b^n] = [a, b]^n [a, b, b]^{n(n-1)/2} [a, b, b, b]^{n(n-1)(n-2)/6},$$

and for $e(G')$ to be small enough to ensure that each $[y_i, y_j]$ has order dividing p^e. It turns out (as a small calculation shows) that everything goes through for the cases $p \ne 3$. However, for the case $p = 3$ we have to add the extra condition that G satisfies the second Engel condition to obtain the required extension.

3. Normal subgroups with cyclic factor-group. The main result of this section is the following:

Theorem 3.1. *Let G be a finite group and K any normal subgroup such that G/K is cyclic. Then*

(i) $|M(G)|$ *divides* $|M(K)|\,|K/K'|$,

(ii) $d(M(G)) \leq d(M(K)) + d(K/K')$.

Proof. Let H be a representing group for G so that H contains a subgroup L in $H' \cap Z(H)$ such that $H/L \cong G$ and $L \cong M(G)$. Choose X in H such that $X/L \cong K$. Since G/K is cyclic, $H = \langle u, X\rangle$ for some element u.

A straightforward argument shows that $H'/X' = \{[u, x]X' \backslash x \in X\}$ so that the map θ defined by $x\theta = [u, x]X'$ is an epimorphism of X onto H'/X'. Clearly LX' is contained in the kernel of θ. We therefore have

(i) $|L/(L \cap X')| = |LX'/X'| \leq |H'/X'| \leq |X/LX'| = |K/K'|$ so that $|M(G)| = |L| \leq |L \cap X'|\,|K/K'|$.

(ii) Since H'/X' is abelian, $d(L/(L \cap X')) \leq d(K/K')$ so that $d(M(G)) = d(L) \leq d(L/(L \cap X')) + d(L \cap X') \leq d(K/K') + d(L \cap X')$.

Since $L \leq Z(X)$, the results follow by the well-known result of Schur (see [1]).

In Theorem 3.1., if K is perfect we have that $|M(G)|$ divides $|M(K)|$ and $d(M(G)) \leq d(M(K))$. If we take $G = S_n$ and $K = A_n$ for $n \geq 8$ it is well known that $|M(G)| = |M(K)| = 2$ and $d(M(G)) = d(M(K)) = 1$. So it can be seen that the results of Theorem 3.1 are, in a sense, best possible.

If G is a finite p-group, another interesting case of Theorem 3.1 is:

Corollary 3.2. *Let G be a finite p-group and K a maximal subgroup of G. Then*

(i) $|M(G)| \leq |M(K)|\,|K/K'|$, *and*

(ii) $d(M(G)) \leq d(M(K)) + d(K)$.

If K is any finite p-group whatsoever, the inequality in part (ii) of Corollary 3.2 becomes equality for the group $K \times Z_p$. Similarly, if K is any finite p-group with K/K' elementary abelian, we have equality in part (i) of Corollary 3.2 for the group $K \times Z_p$.

Corollary 3.3. *Let G be a finite p-group with a maximal subgroup K with trivial multiplicator. Then $M(G)$ is elementary abelian of order at most $p^{d(K)}$.*

Proof. By [4] we have $M(G)^p = 1$ so that $M(G)$ has exponent dividing p.

The result now follows by Corollary 3.2.

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