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THE RANGE OF A VECTOR-VALUED MEASURE

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ABSTRACT. A short proof is given that the weak closure of the range of a totally nonatomic vector-valued measure is convex.

In [3] I. Tweddle showed that the weak closure of the range of a totally nonatomic vector-measure is convex by proving an extension of a lemma used in [1] to establish a generalization of Liapunov's theorem on the range of an atomless measure. Taking Liapunov's theorem in the finite dimensional case the proof can be shortened. The notation is the same as in [3].

Theorem. *Let E be a separated locally convex space, $\mathfrak{M}$ a σ-algebra and $m: \mathfrak{M} \to E$ a totally nonatomic vector-measure. Then*

$$\overline{\operatorname{rg} m}^{\sigma} = \overline{\operatorname{co}}(\operatorname{rg} m) = \bigcap_{X' \in \mathrm{E}'} X'^{-1}(R_{X'}).$$

Here $\operatorname{rg} m = \{m(A) | A \in \mathfrak{M}\}$, $R_{X'} = \operatorname{rg}(X' \cdot m)$ *and*

$$\mathrm{E}' = \left\{ X' | X' = \begin{pmatrix} x'_1 \\ \cdot \\ \cdot \\ \cdot \\ x'_n \end{pmatrix}, x'_i \in E', n \in \mathbf{N} \text{ arbitrary} \right\}.$$

Proof. (i) $\overline{\operatorname{rg} m}^{\sigma} \subset \overline{\operatorname{co}}(\operatorname{rg} m)$ is obvious [2, II. 3.11].

(ii) $R_{X'}$ is convex and compact for all $X' \in E'$ [1, 3. Corollary].

Then $D := \bigcap_{X' \in \mathrm{E}'} X'^{-1}(R_{X'})$ is convex, closed and contains $\operatorname{rg} m$. Hence $\overline{\operatorname{co}}(\operatorname{rg} m) \subset D$.

(iii) Let $x \notin \overline{\operatorname{rg} m}^{\sigma}$; then there is a $\sigma(E, E')$-neighbourhood V of the origin such that $(x + V) \cap \operatorname{rg} m = \emptyset$. By definition of V there exists an $\epsilon > 0$ and continuous linear functionals $x'_1, \cdots, x'_n \in E'$ with $V = \epsilon \cdot \bigcap_{1 \le i \le n} (|x'_i| < 1)$. Let

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$$X' = \begin{pmatrix} x'_1 \\ \cdot \\ \cdot \\ \cdot \\ x'_n \end{pmatrix};$$

then $x \notin X'^{-1}(R_{X'})$ and hence $x \notin D$. Q.E.D.

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