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ON THE EXTREME POINTS OF SOME SETS OF ANALYTIC FUNCTIONS[1]

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ABSTRACT. Let A denote the set of analytic functions defined on the open unit disc. The extreme points of $F = \{f \in A: f(0) = 0$ and $|\mathrm{Re}\, f(z)| < \pi/2\}$ are determined. Also a partial characterization is given for the extreme points of $G_\alpha = \{f \in A: f(0) = 1$ and $|\arg f(z)| < \alpha\pi/2\}$, $0 < \alpha < 1$.

Introduction. The set A of analytic functions defined on the open unit disc U with the topology of uniform convergence on compact subsets of U is a locally convex linear topological space. Recall that $f \in F \subset A$ is called an extreme point of F if f cannot be written as a proper convex combination of two distinct points of F. If F is the set of $f \in A$ such that $f(0) = 1$ and $\mathrm{Re}\, f(z) > 0$, the classical Herglotz formula shows that the extreme points of F are precisely the functions $f(z) = (e^{i\theta} + z)(e^{i\theta} - z)^{-1}$. When F is the set of $f \in A$ such that $|f(z)| \leq 1$, it is known that f is an extreme point of F if and only if $f \in F$ and

$$\int_{-\pi}^{\pi} \log [1 - |f(e^{i\theta})|] d\theta = -\infty \qquad [4, \text{p. } 125].$$

Lately there has been considerable interest in finding the extreme points of certain other subsets of A [1], [2], [3], [5], [6], [7].

In this paper we characterize the extreme points of

$$F = \{f \in A: f(0) = 0 \text{ and } |\mathrm{Re}\, f(z)| < \pi/2\}.$$

We also give a partial characterization for the extreme points of

$$G_\alpha = \{f \in A: f(0) = 1, |\arg f(z)| < \alpha\pi/2\}$$

where $0 < \alpha < 1$.

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D. J. Hallenbeck and T. H. MacGregor [5] have shown that if $f \in H^p$, $p > 1$, and if H denotes the set of functions subordinate to f, then all functions of the form $f \circ \phi$ are extreme points of H, where $\phi(0) = 0$ and ϕ is an inner function. Their proof depends on a theorem due to Ryff [8]. Part of our results follows from the result of Hallenbeck and MacGregor, but we have a direct argument that avoids the use of their result.

Every function in F or G_α, $0 < \alpha < 1$, lies in H^1; in fact, $F \subset H^p$ for all $p < \infty$ and $G_\alpha \subset H^p$ for all $p < 1/\alpha$ [4, pp. 10, 13, 56]. Consequently these functions possess boundary values almost everywhere and can be represented by means of a Poisson integral.

1. Let $S = \{w: |\operatorname{Re} w| < \pi/2\}$ and let F denote the set of analytic functions mapping U into S with $f(0) = 0$. For a function $f \in F$, the boundary values $f(\theta) = \lim_{r \to 1} f(re^{i\theta})$ exist a.e. on $[-\pi, \pi]$. Let $E_f = \{\theta : d(f(\theta), \partial S) > 0\}$, where ∂S denotes the boundary of S and $d(f(\theta), \partial S)$ denotes the distance from $f(\theta)$ to ∂S.

Theorem 1. *f is an extreme point of F if and only if $f \in F$ and the measure of E_f is zero.*

Proof. Let $f \in F$ and suppose E_f has measure zero. Let h be analytic in U with $h(0) = 0$ such that $f \pm h \in F$. In order to show that f is an extreme point of F, it is sufficient to show that $h \equiv 0$. Set

$$g_1(z) = \exp(f(z) + h(z)) \quad \text{and} \quad g_2(z) = \exp(f(z) - h(z)).$$

Then g_1 and g_2 map U into $\{w: e^{-\pi/2} < |w| < e^{\pi/2}\}$. Almost everywhere on $[-\pi, \pi]$, $e^{-\pi/2} \le |g_i(\theta)| \le e^{\pi/2}$ for $i = 1, 2$. Set $E_1 = \{\theta: |\exp f(\theta)| = e^{-\pi/2}\}$ and $E_2 = \{\theta: |\exp f(\theta)| = e^{\pi/2}\}$. Since the boundary values of f and of $f \pm h$ exist a.e., it follows that the boundary values of h exist a.e. If $\theta \in E_1$, then $|\exp h(\theta)| \ge 1$ and $|\exp(-h(\theta))| \ge 1$ a.e. because $e^{-\pi/2} \le |g_i(\theta)|$ a.e. for $i = 1, 2$. Hence $|\exp h(\theta)| = 1$ a.e. on E_1. Similarly $|\exp h(\theta)| = 1$ a.e. on E_2. But, by hypothesis, $E_1 \cup E_2$ has measure 2π and since $\exp h(z)$ is a bounded function it follows that

$$\sup_{|z|<1} |\exp h(z)| = \operatorname{ess\,sup} |\exp h(\theta)| = 1.$$

But $\exp h(0) = 1$, so that $\exp h(z) \equiv 1$ by the maximum modulus theorem. Therefore $h(z) \equiv 0$ and f is an extreme point of F.

Conversely, if E_f has positive measure, there exists ϵ, $0 < \epsilon < 1$, such that $H = \{\theta: d(f(\theta), \partial S) > \epsilon\}$ has positive measure. Let E_1, E_2 be disjoint closed subsets of H of positive measure. Let $\omega_i(z)$ be the harmonic measure of E_i at z. Then $0 < \omega_i(z) < 1$ and if $\theta \in E_i$, $\omega_i(\theta) = 1$ a.e.; if $\theta \notin E_i$, $\omega_i(\theta) = 0$. Set

$$u(z) = \epsilon\{\omega_1(z)/m_1 - \omega_2(z)/m_2\} \min\{m_1, m_2\}$$

where m_i is the measure of E_i. Then $|u(\theta)| \leq \epsilon$ a.e. and if $\theta \notin E_1 \cup E_2$, $u(\theta) = 0$. If $v(z)$ is the harmonic conjugate of $u(z)$ which vanishes at $z = 0$ and if $h(z) = u(z) + iv(z)$,

$$h(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\theta)\, d\theta$$

$$= \frac{\epsilon}{2\pi} \cdot \min\{m_1, m_2\} \cdot \left\{\int_{E_1} \frac{\omega_1(\theta)}{m_1}\, d\theta - \int_{E_2} \frac{\omega_2(\theta)}{m_2}\, d\theta\right\} = 0.$$

If $\theta \in E_1 \cup E_2$, $d(f(\theta), \partial S) > \epsilon$ and $|\mathrm{Re}\, h(\theta)| \leq \epsilon$ a.e. on $E_1 \cup E_2$, so that $|\mathrm{Re}\{f(\theta) \pm h(\theta)\}| \leq \pi/2$ a.e. on $E_1 \cup E_2$. If $\theta \notin E_1 \cup E_2$, then $\mathrm{Re}\, h(\theta) = 0$ and $|\mathrm{Re}\{f(\theta) \pm h(\theta)\}| \leq \pi/2$ a.e. But

$$|\mathrm{Re}\{f(z) \pm h(z)\}| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)|\mathrm{Re}\{f(t) \pm h(t)\}|\, dt \leq \frac{\pi}{2}$$

where $z = re^{i\theta}$ and $P_r(\theta - t)$ is the Poisson kernel. Therefore $f \pm h \in F$ and f is not an extreme point of F since $h \neq 0$.

2. Let $W_\alpha = \{z: |\arg z| < \alpha\pi/2\}$ for $0 < \alpha < 1$ and let G_α denote the set of analytic functions with normalization $f(0) = 1$ which map U into W_α. We need the following lemma before discussing the extreme points of G_α.

Lemma. *If* $z_0 \in W_\alpha$, *then*

$$\sup\{|w|: z_0 \pm w \in W_\alpha\} \leq |z_0| \max\{1, 2\tan(\alpha\pi/2)\}.$$

Proof. For each θ, $0 \leq \theta < \pi$, let 1_θ be the line through z_0 which makes the angle θ with the positive real axis. If $\arg w = \theta = \arg z_0$, then $z_0 \pm w \in W_\alpha$ if and only if $|w| < |z_0|$. If $\theta = \arg w$, $\theta \neq \arg z_0$, let $z_1(\theta)$ be the intersection of 1_θ with

$$\overline{W}_\alpha \cap \{z: \mathrm{Re}\, z \leq \mathrm{Re}\, z_0\}$$

and let $z_2(\theta)$ be the intersection of 1_θ with

$$\overline{W}_\alpha \cap \{z: \mathrm{Re}\, z \geq \mathrm{Re}\, z_0\}.$$

Here $\overline{W}_\alpha$ denotes the closure of W_α. (For $\theta = \pi/2$, choose either point for

$z_1(\theta)$, the other for $z_2(\theta)$.) Then $z_0 \pm w \in W_\alpha$ if and only if

$$|w| < \min\{|z_0 - z_1(\theta)|, |z_0 - z_2(\theta)|\} \le |z_0 - z_1(\theta)|.$$

Thus

$$\sup\{|w|: z_0 \pm w \in W_\alpha\} \le \sup\{|z_0 - z|: z \in \partial W_\alpha, \operatorname{Re} z \le \operatorname{Re} z_0\}$$

$$= \max\{|z_0|, |z_0 - z_1(\pi/2)|, |z_0 - z_2(\pi/2)|\}.$$

But it is easily seen that $|z_0 - z_i(\pi/2)| \le 2|z_0|\tan(\alpha\pi/2)$ and the lemma follows.

Theorem 2. *Let $f \in G_\alpha$ and $E_f = \{\theta: d(f(\theta), \partial W_\alpha) > 0\}$. If E_f has measure zero, then f is an extreme point of G_α.*

Proof. Suppose h is analytic on U, $h(0) = 0$, and $f \pm h \in G_\alpha$. By the lemma, $|h(z)| \le M|f(z)|$ where $M = \max\{1, 2\tan(\alpha\pi/2)\}$ and hence $h(z)/f(z)$ is a bounded analytic function. Suppose $h(z)$ is not identically zero. Since $f(\theta) \in \partial W_\alpha$ a.e. and $f(\theta) \pm h(\theta) \in \overline{W}_\alpha$ a.e., $\arg h(\theta) = \arg f(\theta) + k(\theta)\pi$ a.e. where $k(\theta) \in \mathbf{Z}$. Since $f(\theta)$ is nonzero a.e., $h(\theta)/f(\theta)$ is defined a.e. and the above shows that $h(\theta)/f(\theta)$ is real a.e. Since $h(z)/f(z)$ is a bounded function with real boundary values a.e., $h(z)/f(z)$ is everywhere real and is therefore a constant. Since $h(0) = 0$, this constant is zero and h is identically zero. Therefore f is an extreme point of F.

The method used in the above proof easily yields the following generalization.

Let K be a convex polygon (not necessarily compact) which is contained in some wedge W_α, $0 < \alpha < 1$. Let F denote the set of analytic functions mapping U into K and normalized such that $f(0) = z_0$, where z_0 is an interior point of K. If $f \in F$ and $\{\theta: d(f(\theta), \partial K) > 0\}$ has measure zero, then f is an extreme point of F.

In the direction of the converse of Theorem 2 we can only give a partial result. However this does not depend on the fact that $0 < \alpha < 1$.

Theorem 3. *Let $f \in G_\alpha$ and suppose there is an interval $I = \{\theta: \gamma < \theta < \beta\}$ with the property that $d(f(\theta), \partial W_\alpha) > \epsilon > 0$ a.e. on I and $\operatorname{ess}_I \sup|f(\theta)| \le M$. Then f is not an extreme point of G_α.*

Proof. For $\beta \le \theta \le 2\pi + \gamma$, define $u(\theta) = -\theta$ and for $\gamma \le \theta \le \beta$, let $u(\theta)$ be a C^1 function with $u(\beta) = -\beta$, $u(\gamma) = -2\pi - \gamma$ and $u'(\beta) = u'(\gamma) = -1$. Then u is continuously differentiable on $[\gamma, 2\pi + \gamma]$. For $z = re^{i\theta}$, set

$$u(z) = \frac{1}{2\pi}\int_{\gamma}^{2\pi+\gamma} P_r(\theta - t)u(t)\,dt$$

and let $v(z)$ be the harmonic conjugate of u vanishing at $z = 0$. Then u and v are both continuously extendible to $|z| \le 1$ [4, p. 83]. Set

$$h_1(z) = B^{-1}\exp\{iu(z) - v(z)\}$$

where $B = \max_{|z|\le 1}|\exp\{iu(z) - v(z)\}|$ and set

$$h(z) = (\epsilon_0/(M+1)) \cdot zf(z) \cdot h_1(z)$$

where $\epsilon_0 = \min\{1, \epsilon\}$. Almost everywhere on I, $d(f(\theta), \partial W_\alpha) > \epsilon$ and $|h(\theta)| < \epsilon$, so $f(\theta) \pm h(\theta) \in \overline{W}_\alpha$: For $\theta \notin I$, $\arg h(\theta) = \arg f(\theta)$ a.e., and $|h(\theta)| \le |f(\theta)|$ a.e., so $f(\theta) \pm h(\theta) \in \overline{W}_\alpha$. The Poisson integral representation shows that $\mathrm{Re}\{f(z) \pm h(z)\} > 0$ so that a single-valued branch of $\arg\{f(z) \pm h(z)\}$ can be chosen. Choosing the branch so that $\arg 1 = 0$ and using the Poisson representation, it then follows that

$$|\arg f(z) \pm h(z)| < \alpha\pi/2 \quad \text{for } z \in U.$$

Hence $f \pm h \in G_\alpha$ and f is not an extreme point of G_α.

In [1], D. A. Brannan, J. G. Clunie, and W. E. Kirwan have shown that the extreme points of G_α, $\alpha > 1$, are precisely the functions

$$f_x(z) = ((1 + xz)/(1 - xz))^\alpha, \qquad |x| = 1.$$

When $0 < \alpha < 1$, these functions are extreme points of G_α but they do not exhaust the set of extreme points as Theorem 2 shows.

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