PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 2, August 1974

THE *DAD* THEOREM FOR ARBITRARY ROW SUMS[1]

RICHARD A. BRUALDI

ABSTRACT. Given an $m \times m$ symmetric nonnegative matrix A and a positive vector $R = (r_1, \cdots, r_m)$, necessary and sufficient conditions are obtained in order that there exist a diagonal matrix D with positive main diagonal such that DAD has row sum vector R.

A nonnegative $m \times n$ matrix A is called *completely decomposable* if there exist partitions α_1, α_2 of $\{1, \cdots, m\}$ and β_1, β_2 of $\{1, \cdots, n\}$ into nonvacuous sets such that $A[\alpha_1, \beta_2]$ and $A[\alpha_2, \beta_1]$ are zero matrices. Here we use the notation that $A[\alpha, \beta]$ is the submatrix of A whose rows are indexed by α and whose columns are indexed by β, the rows and columns in $A[\alpha, \beta]$ appearing in the same order as in A. If $m = n$, the matrix A is called *completely reducible* if there exists a partition α_1, α_2 of $\{1, \cdots, m\}$ into nonvacuous sets such that $A[\alpha_1, \alpha_2]$ and $A[\alpha_2, \alpha_1]$ are zero matrices.

Generalizing theorems of Sinkhorn and Knopp [10] and Brualdi, Parter, and Schneider [1], Menon [7] proved the following theorem: Let A be an $m \times n$ nonnegative matrix and let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be positive vectors with $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. Let $\mathfrak{A}(R, S)$ denote the class of all $m \times n$ nonnegative matrices with row sum vector R and column sum vector S. Then there exist diagonal matrices D_1 and D_2 with positive main diagonals such that D_1AD_2 is in $\mathfrak{A}(R, S)$ if and only if there is a matrix in $\mathfrak{A}(R, S)$ which has the same zero pattern as A. (We say that a matrix B has the *same zero pattern* as A provided $b_{ij} = 0$ if and only if $a_{ij} = 0$.) If, in addition, A is not completely decomposable, the diagonal matrices D_1, D_2 are unique up to positive scalar factor: if U_1AU_2 is in $\mathfrak{A}(R, S)$ then there exists $\delta > 0$ such that $U_1 = \delta D_1$, $U_2 = \delta^{-1} D_2$. Brualdi [2] proved that given A there exists a matrix in $\mathfrak{A}(R, S)$ with the same zero pattern as A if and only if the following condition is satisfied.

Received by the editors October 23, 1973.

AMS (MOS) subject classifications (1970). Primary 15A21, 15A48; Secondary 15A51.

Key words and phrases. Nonnegative matrix, diagonal matrix, symmetric matrix, completely reducible, completely decomposable, row sum vector.

[1] Research partially supported by NSF Grant No. 37978X.

(1) *For all partitions* α_1, α_2 *of* $\{1, \cdots, m\}$ *and* β_1, β_2 *of* $\{1, \cdots, n\}$ *into nonempty sets such that* $A[\alpha_1, \beta_2]$ *is a zero matrix,* $\sum_{j\epsilon\beta_1} s_j \geq \sum_{i\epsilon\alpha_1} r_i$ *with equality holding if and only if* $A[\alpha_2, \beta_1]$ *is also a zero matrix.*

These two results when combined yield the following theorem.

Theorem 1. *Let* A *be an* $m \times n$ *nonnegative matrix and* $R = (r_1, \cdots, r_m)$ *and* $S = (s_1, \cdots, s_n)$ *positive vectors with* $r_1 + \cdots + r_m = s_1 + \cdots + s_n$. *There exist diagonal matrices* D_1, D_2 *with positive main diagonals such that* $D_1 A D_2$ *is in* $\mathfrak{A}(R, S)$ *if and only if* (1) *is satisfied. The diagonal matrices* D_1, D_2 *are unique up to scalar factor if* A *is not completely decomposable.*

An alternate derivation of this result is given by Menon and Schneider [8]; a recent slick derivation of all but the uniqueness part is given by Sinkhorn [11]. A special case of the above theorem which is of interest is the case of doubly stochastic matrices ($m = n$, $R = (1, \cdots, 1)$, $S = (1, \cdots, 1)$). This case had been previously settled by Sinkhorn and Knopp [10] and Brualdi, Parter, Schneider [1].

If $m = n$ and A is a symmetric matrix one naturally wonders whether the diagonal matrices D_1, D_2 in Theorem 1 can be taken to be equal. For the doubly stochastic case partial results have been obtained by Marcus and Newman [5], Brualdi, Parter, and Schneider [1] and Marshall and Olkin [6]. A complete answer for the doubly stochastic case was given by Csima and Datta [4] who proved the following theorem.

Theorem 2. *Let* A *be a symmetric nonnegative matrix. There exists a diagonal matrix* D *with positive main diagonal such that* DAD *is a (symmetric) doubly stochastic matrix if and only if there is a symmetric doubly stochastic matrix with the same zero pattern as* A.

Csima and Datta actually state their theorem in terms of total support. A nonnegative square matrix has total support if it has at least one positive entry and each positive entry lies on some positive diagonal. The zero patterns of doubly stochastic matrices are precisely the zero patterns of matrices with total support [9].

Our purpose here is to extend Theorem 2 to arbitrary row sum vectors. For a positive vector $R = (r_1, \cdots, r_m)$ let $\mathfrak{A}(R)$ denote the class of all $m \times m$ symmetric nonnegative matrices with row sum vector (and thus column sum vector) equal to R. In [3] we characterized the zero patterns of matrices in $\mathfrak{A}(R)$ by the following theorem.

Theorem 3. *Let* $R = (r_1, \cdots, r_m)$ *be a positive vector and let* A *be an*

$m \times m$ nonnegative matrix. There exists a matrix in $\mathfrak{A}(R)$ with the same zero pattern as A if and only if the following condition is satisfied.

(2) *For all partitions α, β, γ of $\{1, \cdots, m\}$ such that $A[\beta \cup \gamma, \gamma]$ is a zero matrix, $\sum_{i \epsilon \alpha} r_i \geq \sum_{j \epsilon \gamma} r_j$.*

Equality is to occur if and only if $A[\alpha, \alpha \cup \beta]$ is also a zero matrix.

We shall prove the following theorem which generalizes Theorem 2.

Theorem 4. *Let $R = (r_1, \cdots, r_m)$ be a positive vector and A an $m \times m$ symmetric nonnegative matrix. There exists a diagonal matrix D with positive main diagonal such that DAD is in $\mathfrak{A}(R)$ if and only if (2) is satisfied.*

We require two lemmas.

Lemma 5. *Let $R = (r_1, \cdots, r_m)$ be a positive vector and let A be an $m \times m$ symmetric nonnegative matrix satisfying (2) which is not completely decomposable. Then there exists a diagonal matrix D with positive main diagonal such that DAD is in $\mathfrak{A}(R)$.*

Proof. If A satisfies (2), then according to Theorem 3 there exists a matrix in $\mathfrak{A}(R)$ with the same zero pattern as A. From Menon's theorem we conclude that there exist diagonal matrices D_1 and D_2 with positive main diagonals such that D_1AD_2 is in $\mathfrak{A}(R, R)$ with D_1, D_2 uniquely determined up to scalar factor. Then $(D_1AD_2)^t = D_2^tA^tD_1^t = D_2AD_1$ is also in $\mathfrak{A}(R, R)$. Thus $D_2 = \delta D_1$ so that with $D = (\delta)^{1/2}D_1$, DAD is in $\mathfrak{A}(R, R)$. Since DAD is a symmetric matrix, DAD is in $\mathfrak{A}(R)$ and the lemma is proved.

Lemma 6. *Let A be an $m \times m$ symmetric nonnegative matrix which is not completely reducible. Then either A is not completely decomposable or else there exists a permutation matrix P such that PAP^t has the form*

$$\begin{bmatrix} 0 & 0 & B_1 \\ 0 & A_1 & 0 \\ B_1^t & 0 & 0 \end{bmatrix} \tag{3}$$

where B_1 is a nonvacuous matrix which is not completely decomposable and A_1 is a symmetric matrix which is not completely reducible.

Proof. Suppose A is completely decomposable. Then there exist partitions α_1, α_2 and β_1, β_2 of $\{1, \cdots, m\}$ into nonempty sets such that $A[\alpha_1, \beta_2]$ and $A[\alpha_2, \beta_1]$ are zero matrices. Define a partition α, β, γ

of $\{1, \cdots, m\}$ by $\alpha = \beta_1 \backslash \alpha_1$, $\gamma = \alpha_1 \backslash \beta_1$ and $\beta = \{1, \cdots, m\} \backslash (\alpha \cup \gamma)$. Suppose $\beta_1 \subseteq \alpha_1$. Then β_1, β_2 is a partition of $\{1, \cdots, m\}$ into nonempty sets with $A[\beta_1, \beta_2]$ a zero matrix since $\beta_1 \subseteq \alpha_1$. Since A is symmetric, $A[\beta_2, \beta_1]$ is also a zero matrix and this contradicts the fact that A is not completely reducible. Suppose $\alpha_1 \subseteq \beta_1$, then α_1, α_2 is a partition of $\{1, \cdots, m\}$ into nonempty sets with $A[\alpha_2, \alpha_1]$ a zero matrix. Since A is symmetric, $A[\alpha_1, \alpha_2]$ is also a zero matrix and we have a similar contradiction. We conclude that α and γ as defined above are nonempty.

But now $A[\gamma, \gamma]$ is a zero matrix since $\gamma \subseteq \alpha_1$ and $\gamma \cap \beta_1 = \emptyset$ so that $\gamma \subseteq \beta_2$. We verify $A[\beta, \gamma]$ is a zero matrix by first observing that $\beta = (\{1, \cdots, m\} \backslash (\alpha_1 \cup \beta_1)) \cup (\alpha_1 \cap \beta_1)$. Then $A[\alpha_1 \cap \beta_1, \gamma]$ is a zero matrix since $\alpha_1 \cap \beta_1 \subseteq \alpha_1$ and $\gamma \subseteq \beta_2$. Also $A[\{1, \cdots, m\} \backslash (\alpha_1 \cup \beta_1), \gamma]$ is a zero matrix, since $\{1, \cdots, m\} \backslash (\alpha_1 \cup \beta_1) \subseteq \beta_2$ and $\gamma \subseteq \alpha_1$ and since $A[\beta_2, \alpha_1]$ is a zero matrix by the symmetry of A.

Let $R = (r_1, \cdots, r_m)$ be the row (and column) sum vector of A. Since $A[\alpha_1, \beta_2]$ and $A[\alpha_2, \beta_1]$ are zero matrices, $\sum_{i \in \alpha_1} r_i = \sum_{j \in \beta_1} r_j$ and thus $\sum_{i \in \alpha_1 \backslash \beta_1} r_i = \sum_{j \in \beta_1 \backslash \alpha_1} r_j$ or equivalently $\sum_{i \in \alpha} r_i = \sum_{j \in \gamma} r_j$. Since A is a nonnegative matrix this implies that $A[\alpha, \alpha]$, $A[\alpha, \beta]$ and thus $A[\beta, \alpha]$ by symmetry are zero matrices. Thus, there exists a permutation matrix P so that

$$PAP^t = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} \begin{array}{ccc} \alpha & \beta & \gamma \end{array} & \\ \begin{bmatrix} 0 & 0 & B_1 \\ 0 & A_1 & 0 \\ B_1^t & 0 & 0 \end{bmatrix} & \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \end{array}$$

Suppose B_1 were completely decomposable. Then there exist partitions α', α'' of α and γ', γ'', of γ into nonempty sets such that $B_1[\alpha', \gamma'']$ and $B_1[\alpha'', \gamma']$ are zero matrices. But then $\pi_1 = \alpha' \cup \gamma'$, $\pi_2 = \{1, \cdots, m\} \backslash (\alpha' \cup \gamma')$ is a partition of $\{1, \cdots, m\}$ into nonempty sets with $A[\pi_1, \pi_2]$ and $A[\pi_2, \pi_1]$ zero matrices contradicting the fact that A is not completely reducible. Thus B_1 is not completely decomposable. If A_1 were nonvacuous and completely reducible, then A would be completely reducible. The proof of the lemma is now complete.

Corollary 7. *Let A be an $m \times m$ symmetric nonnegative matrix. Then either A is not completely decomposable or else there exists a permutation matrix P such that PAP^t has the form*

$$\begin{bmatrix} & & & & & & B_1 \\ & 0 & & & & \cdot^{\cdot^{\cdot}} & \\ & & & & B_k & & \\ & & & B_{k+1} & & & \\ & & B_k^t & & & & \\ & \cdot^{\cdot^{\cdot}} & & & & 0 & \\ B_1^t & & & & & & \end{bmatrix} \tag{4}$$

where $B_1, \cdots, B_{k+1}$ *are not completely decomposable with* B_{k+1} *a possibly vacuous symmetric matrix.*

The corollary follows by induction from the lemma.

Proof of Theorem 4. Let A be an $m \times m$ symmetric nonnegative matrix. Suppose there is a diagonal matrix D with positive main diagonal with DAD in $\mathfrak{A}(R)$. Then according to Theorem 3 (and indeed it is easy to verify) (2) is satisfied.

Now suppose (2) is satisfied. First assume that A is not completely reducible. If, in addition, A is not completely decomposable, the existence of the required D is obtained from Lemma 5. If, on the other hand, A is completely decomposable, there exists according to Corollary 7 a permutation matrix P such that PAP^t has the form (4) where $B_1, \cdots, B_{k+1}$ are not completely decomposable. We may assume without loss in generality that A itself has the form (4). Thus there exists a partition $\pi_1, \cdots, \pi_k, \pi_{k+1}, \rho_k, \cdots, \rho_1$ of $\{1, \cdots, m\}$ such that

$$A[\pi_1, \rho_1] = B_1, \cdots, A[\pi_k, \rho_k] = B_k, \; A[\pi_{k+1}, \pi_{k+1}] = B_{k+1}.$$

Let R_{π_s} be the vector $(r_i\colon i \in \pi_s)$, $s = 1, \cdots, k+1$, and R_{ρ_t} be the vector $(r_i\colon i \in \rho_t)$, $t = 1, \cdots, k$, with the components occuring in the same order as in R. It is easy to check that B_j satisfies the hypothesis of Theorem 1 with respect to the vectors R_{π_j} and R_{ρ_j} $(j = 1, \cdots, t)$ and that B_{k+1} satisfies the conditions of Theorem 3 with respect to the vector $R_{\pi_{k+1}}$. Thus by Theorem 1 there exist diagonal matrices $D_1, \cdots, D_k, E_1, \cdots, E_k$ with positive main diagonals such that $D_j B_j E_j$ is in $\mathfrak{A}(R_{\pi_j}, R_{\rho_j})$, $j = 1, \cdots, k$. By Lemma 5 there exists a diagonal matrix D_{k+1} with positive main diagonal such that $D_{k+1} A_{k+1} D_{k+1}$ is in $\mathfrak{A}(R_{\pi_{k+1}})$. If we let D be the diagonal matrix given in block form by

$$D = \operatorname{diag}(D_1, \cdots, D_k, D_{k+1}, E_k, \cdots, E_1),$$

Then DAD is in $\mathfrak{A}(R)$.

If A were not completely reducible, then there exists a permutation matrix Q such that QAQ^t is the direct sum of symmetric matrices $C_1, \cdots, C_l$ which are not completely reducible. Applying the first part of the proof to the C_i and extending to A in the obvious way, we complete the proof of the theorem.

REFERENCES

1. R. A. Brualdi, S. V. Parter and H. Schneider, *The diagonal equivalence of a nonnegative matrix to a stochastic matrix*, J. Math. Anal. Appl. **16** (1966), 31–50. MR **34** #5844.

2. R. A. Brualdi, *Convex sets of non-negative matrices*, Canad. J. Math. **20** (1968), 144–157. MR **36** #2636.

3. ———, *Combinatorial properties of symmetric nonnegative matrices*, Proc. Internat. Conf. 'Combinatorial Theories' held in Rome, Sept. 3–15, 1973 (to appear).

4. J. Csima and B. N. Datta, *The DAD theorem for symmetric nonnegative matrices*, J. Combinatorial Theory Ser. A **12** (1972), 147–152. MR **44** #6726.

5. M. Marcus and H. Minc, *A survey of matrix theory and matrix inequalities*, Allyn and Bacon, Boston, Mass., 1964, p. 130. MR **29** #112.

6. A. W. Marshall and I. Olkin, *Scaling of matrices to achieve specified row and column sums*, Numer. Math. **12** (1968), 83–90. MR **39** #235.

7. M. V. Menon, *Matrix links, an extremization problem, and the reduction of a non-negative matrix to one with prescribed row and column sums*, Canad. J. Math. **20** (1968), 225–232. MR **36** #3804.

8. M. V. Menon and H. Schneider, *The spectrum of a nonlinear operator associated with a matrix*, Linear Algebra and Appl. **2** (1969), 321–334. MR **40** #162.

9. H. Perfect and L. Mirsky, *The distribution of positive elements in doubly-stochastic matrices*, J. London Math. Soc. **40** (1965), 689–698. MR **32** #503.

10. R. Sinkhorn and P. Knopp, *Concerning nonnegative matrices and doubly stochastic matrices*, Pacific J. Math. **21** (1967), 343–348. MR **35** #1617.

11. R. Sinkhorn, *Diagonal equivalence to matrices with prescribed row and column sums*. II, Proc. Amer. Math. Soc. 45 (1974), 195–198.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706